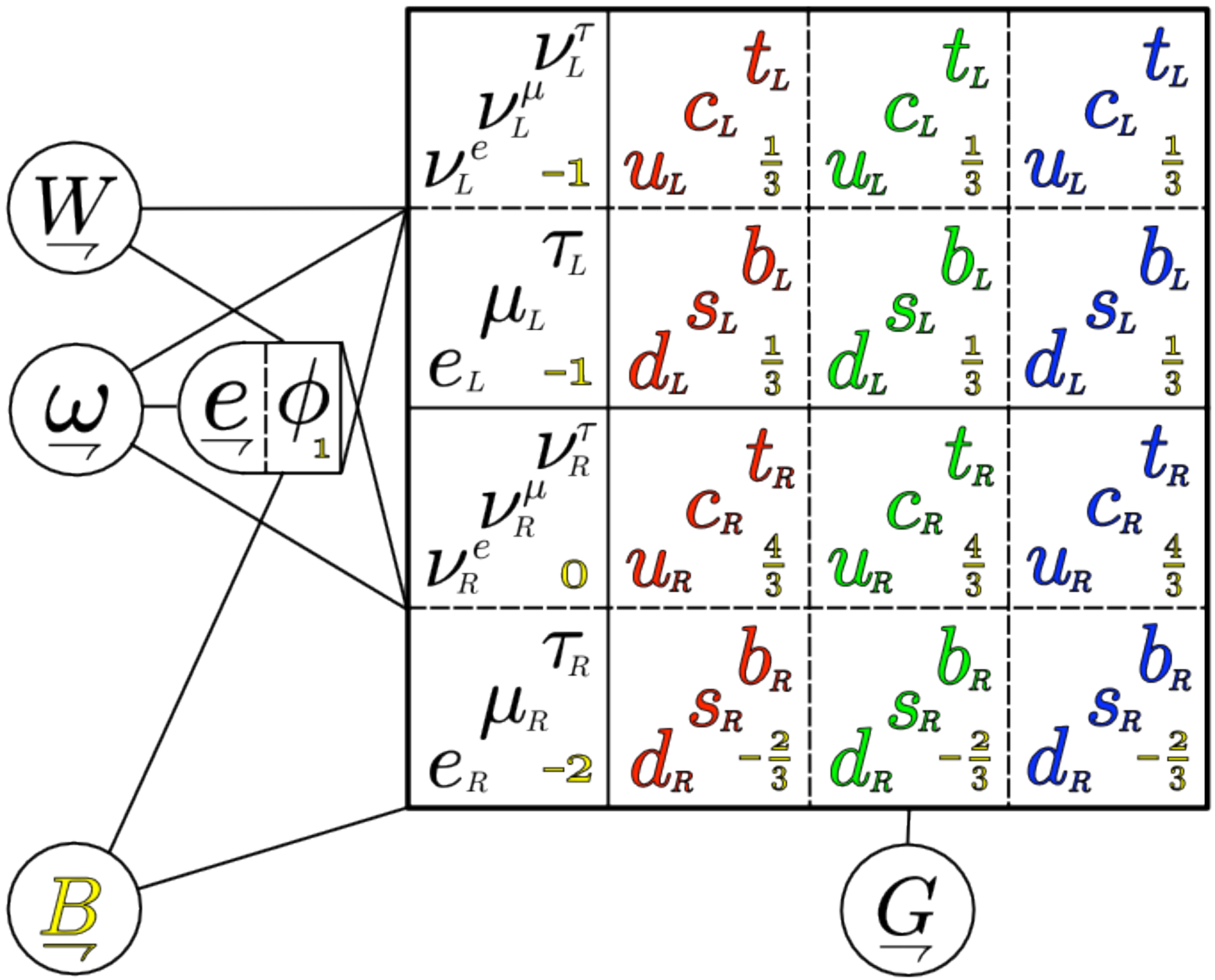
Standard model and gravity



http://deferentialgeometry.org

Garrett Lisi

FQXi

Everything as a principal bundle connection

$$1918,\ \text{Weyl}: \quad \underrightarrow{A} \in \underrightarrow{\text{Lie}}(G)$$

$$1954,\ \text{Y.M.}: \quad \underrightarrow{A} = \underrightarrow{B} + \underrightarrow{W} + \underrightarrow{G} \;\in\; \underrightarrow{\text{Lie}}(G) = \underrightarrow{su}(1) + \underrightarrow{su}(2) + \underrightarrow{su}(3)$$

$$1967,\ \text{F.P.}: \quad \underset{\cdot\to}{A} = \underrightarrow{A} + \underset{\cdot}{g} \;\in\; \underset{\cdot\to}{\text{Lie}}(G)$$

$$1977,\ \text{M.M.}: \quad \underrightarrow{A} = \underrightarrow{\omega} + \underrightarrow{e} \;\in\; \underrightarrow{\text{Lie}}(G) = \underrightarrow{so}(1,4)$$

$$2002,\ \text{Y.T.}: \quad \underset{\cdot}{\psi} = \underset{\cdot}{g}$$

$$2005,\ \text{Y.T.}: \quad \underset{\cdot\to}{A} = \tfrac{1}{2}\underrightarrow{\omega} + \tfrac{1}{4}\underrightarrow{e}\,\phi + \underrightarrow{B} + \underrightarrow{W} + \underrightarrow{G} + \underset{\cdot}{\nu}^{e} + \underset{\cdot}{e} + \underset{\cdot}{u} + \underset{\cdot}{d}$$
$$\in\; \underrightarrow{\text{Lie}}(G) = \underrightarrow{Cl}(1,7)$$

$$\text{now},\ \text{Y.T.}: \quad \underset{\cdot\to}{A} = \tfrac{1}{2}\underrightarrow{\omega} + \tfrac{1}{4}\underrightarrow{e}\,\phi + \underrightarrow{B} + \underrightarrow{W} + \underrightarrow{G} + \underset{\cdot}{\nu}^{e} + \underset{\cdot}{e} + \underset{\cdot}{u} + \underset{\cdot}{d}$$
$$+ \underset{\cdot}{\nu}^{\mu} + \underset{\cdot}{\mu} + \underset{\cdot}{c} + \underset{\cdot}{s} + \underset{\cdot}{\nu}^{\tau} + \underset{\cdot}{\tau} + \underset{\cdot}{t} + \underset{\cdot}{b}$$
$$\in\; \underrightarrow{\text{Lie}}(G) = \underrightarrow{e8}?$$

Standard model and gravity in a matrix

$$A = H + G + \psi = \begin{bmatrix} H^+ & \psi^- \\ & G^- \end{bmatrix} \in so(1,7) + so(8) + \mathbb{C}(8\times 8)$$

$$= \begin{bmatrix}
\frac{1}{2}\omega_L + iW^3 & iW^1 + W^2 & -\frac{1}{4}e_R\phi_0^* & \frac{1}{4}e_R\phi_+ & \nu_L & u^r_L & u^g_L & u^b_L \\
iW^1 - W^2 & \frac{1}{2}\omega_L - iW^3 & \frac{1}{4}e_R\phi_+^* & \frac{1}{4}e_R\phi_0 & e_L & d^r_L & d^g_L & d^b_L \\
-\frac{1}{4}e_L\phi_0 & \frac{1}{4}e_L\phi_+ & \frac{1}{2}\omega_R + iB & & \nu_R & u^r_R & u^g_R & u^b_R \\
\frac{1}{4}e_L\phi_+^* & \frac{1}{4}e_L\phi_0^* & & \frac{1}{2}\omega_R - iB & e_R & d^r_R & d^g_R & d^b_R \\
 & & & & iB & & & \\
 & & & & & \frac{-i}{3}B + iG^{3+8} & iG^1 - G^2 & iG^4 - G^5 \\
 & & & & & iG^1 + G^2 & \frac{-i}{3}B - iG^{3+8} & iG^6 - G^7 \\
 & & & & & iG^4 + G^5 & iG^6 + G^7 & \frac{-i}{3}B - \frac{2i}{\sqrt{3}}G^8
\end{bmatrix}$$

Correct interactions and charges from **curvature**:

$$F = dA + AA$$

$$= (dH + HH) + (dG + GG) + (d\psi + H\psi + \psi G)$$

Gravitational part of the connection

Using **chiral** (*Weyl*) $\mathbb{C}(4\times 4)$ representation of **Cl(1,3) Dirac matrices**:

$$\gamma_0 = \sigma_1 \otimes 1 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \qquad \gamma_\pi = i\sigma_2 \otimes \sigma_\pi = \begin{bmatrix} & \sigma_\pi \\ -\sigma_\pi & \end{bmatrix}$$

$$\gamma_{0\varepsilon} = \gamma_0\gamma_\varepsilon = \begin{bmatrix} -\sigma_\varepsilon & \\ & \sigma_\varepsilon \end{bmatrix} \qquad \gamma_{\varepsilon\pi} = \gamma_\varepsilon\gamma_\pi = \begin{bmatrix} -i\epsilon_{\varepsilon\pi\tau}\sigma_\tau & \\ & -i\epsilon_{\varepsilon\pi\tau}\sigma_\tau \end{bmatrix}$$

Spacetime frame and **spin connection**:

$$\underset{\rightharpoondown}{\omega} + \underset{\rightharpoondown}{e} = \underset{\rightharpoondown}{dx}^a \tfrac{1}{2}\omega_a{}^{\mu\nu}\gamma_{\mu\nu} + \underset{\rightharpoondown}{dx}^a (e_a)^\mu \gamma_\mu$$

$$= \begin{bmatrix} (-\underset{\rightharpoondown}{\omega}^{0\varepsilon}\sigma_\varepsilon - \frac{i}{2}\underset{\rightharpoondown}{\omega}^{\varepsilon\pi}\epsilon_{\varepsilon\pi\tau}\sigma_\tau) & (\underset{\rightharpoondown}{e}^0 + \underset{\rightharpoondown}{e}^\pi\sigma_\pi) \\ (\underset{\rightharpoondown}{e}^0 - \underset{\rightharpoondown}{e}^\pi\sigma_\pi) & (\underset{\rightharpoondown}{\omega}^{0\varepsilon}\sigma_\varepsilon - \frac{i}{2}\underset{\rightharpoondown}{\omega}^{\varepsilon\pi}\epsilon_{\varepsilon\pi\tau}\sigma_\tau) \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\rightharpoondown}{\omega}_L & \underset{\rightharpoondown}{e}_R \\ \underset{\rightharpoondown}{e}_L & \underset{\rightharpoondown}{\omega}_R \end{bmatrix} \quad \in \quad \underset{\rightharpoondown}{Cl}^{1+2}(1,3)$$

Note algebraic equivalence: $Cl^{1+2}(1,3) = Cl^2(1,4) = so(1,4)$

Bosonic part of the connection

$$\underrightarrow{H} = \tfrac{1}{2}\underrightarrow{\omega} + \tfrac{1}{4}\underrightarrow{e}\,\phi + \underrightarrow{B} + \underrightarrow{W} = \begin{bmatrix} \frac{1}{2}\omega_{\underrightarrow{L}} + i\underrightarrow{W}^3 & i\underrightarrow{W}^1 + \underrightarrow{W}^2 & -\frac{1}{4}e_{\underrightarrow{R}}\phi_0^* & \frac{1}{4}e_{\underrightarrow{R}}\phi_+ \\ i\underrightarrow{W}^1 - \underrightarrow{W}^2 & \frac{1}{2}\omega_{\underrightarrow{L}} - i\underrightarrow{W}^3 & \frac{1}{4}e_{\underrightarrow{R}}\phi_+^* & \frac{1}{4}e_{\underrightarrow{R}}\phi_0 \\ -\frac{1}{4}e_{\underrightarrow{L}}\phi_0 & \frac{1}{4}e_{\underrightarrow{L}}\phi_+ & \frac{1}{2}\omega_{\underrightarrow{R}} + i\underrightarrow{B} & \\ \frac{1}{4}e_{\underrightarrow{L}}\phi_+^* & \frac{1}{4}e_{\underrightarrow{L}}\phi_0^* & & \frac{1}{2}\omega_{\underrightarrow{R}} - i\underrightarrow{B} \end{bmatrix}$$

$$= d\underrightarrow{x}^a \tfrac{1}{2} h_a{}^{\alpha\beta}\gamma_{\alpha\beta} \quad \in \quad \underrightarrow{so}(1,7) = \underrightarrow{Cl}^2(1,7) \subset \underrightarrow{\mathbb{C}}(8\times 8)$$

Clifford bivector parts:

$$\underrightarrow{\omega} = d\underrightarrow{x}^a \tfrac{1}{2}\omega_a{}^{\mu\nu}\gamma_{\mu\nu} \qquad \leftarrow \text{spin connection}$$

$$\underrightarrow{e}\,\phi = d\underrightarrow{x}^a (e_a)^\mu \phi^\phi \gamma_{\mu\phi} \quad \begin{cases} \underrightarrow{e} = d\underrightarrow{x}^a (e_a)^\mu \gamma_\mu & \leftarrow \text{frame (vierbein)} \\ \phi = \phi^\phi \gamma_\phi \quad \left\{ \begin{aligned} \phi_+ &= (-\phi^5 + i\phi^6) \\ \phi_0 &= (\phi^7 + i\phi^8) \end{aligned} \right\} & \leftarrow \text{Higgs} \quad \phi\phi = -M^2 \end{cases}$$

$$\underrightarrow{B} = -d\underrightarrow{x}^a \tfrac{1}{2} B_a \big(\gamma_{56} - \gamma_{78}\big) \qquad \leftarrow\downarrow \text{electroweak gauge fields}$$

$$\underrightarrow{W} = -\tfrac{1}{2}\underrightarrow{W}^1\big(\gamma_{67} + \gamma_{58}\big) - \tfrac{1}{2}\underrightarrow{W}^2\big(-\gamma_{57} + \gamma_{68}\big) - \tfrac{1}{2}\underrightarrow{W}^3\big(\gamma_{56} + \gamma_{78}\big)$$

indices: $\quad 0 \le a, b \le 3 \qquad 0 \le \mu, \nu \le 3 \qquad 5 \le \phi, \psi \le 8$

Curvature of bosonic part

$$\underrightarrow{\underrightarrow{F}} = \underrightarrow{d}\,\underrightarrow{H} + \underrightarrow{H}\,\underrightarrow{H} \qquad\qquad \underrightarrow{H} = \tfrac{1}{2}\underrightarrow{\omega} + \tfrac{1}{4}\underrightarrow{e}\,\phi + \underrightarrow{B} + \underrightarrow{W}$$

$$= \Big(\tfrac{1}{2}(\underrightarrow{d}\,\underrightarrow{\omega} + \tfrac{1}{2}\underrightarrow{\omega}\,\underrightarrow{\omega}) + \tfrac{1}{16}M^2\underrightarrow{e}\,\underrightarrow{e}\Big) \qquad \leftarrow \text{spacetime } \gamma_{\mu\nu}$$

$$+ \Big(\tfrac{1}{4}(\underrightarrow{d}\,\underrightarrow{e} + \tfrac{1}{2}[\underrightarrow{\omega}, \underrightarrow{e}])\phi - \tfrac{1}{4}\underrightarrow{e}\big(\underrightarrow{d}\,\phi + [\underrightarrow{B} + \underrightarrow{W}, \phi]\big)\Big) \qquad \leftarrow \text{mixed } \gamma_{\mu\phi}$$

$$+ \Big(\underrightarrow{d}\,\underrightarrow{B} + \underrightarrow{d}\,\underrightarrow{W} + \underrightarrow{W}\,\underrightarrow{W}\Big) \qquad \leftarrow \text{higher } \gamma_{\phi\psi}$$

$$= \tfrac{1}{2}\big(\underrightarrow{\underrightarrow{R}} + \tfrac{1}{8}M^2\underrightarrow{e}\,\underrightarrow{e}\big) + \tfrac{1}{4}\big(\underrightarrow{\underrightarrow{T}}\phi - \underrightarrow{e}\,\underrightarrow{D}\phi\big) + \big(\underrightarrow{\underrightarrow{F}}_B + \underrightarrow{\underrightarrow{F}}_W\big)$$

$$= \underrightarrow{\underrightarrow{F}}_s + \underrightarrow{\underrightarrow{F}}_m + \underrightarrow{\underrightarrow{F}}_h$$

Modified BF action over 4D base **manifold**:

$$S = \int \big\langle \underrightarrow{\underrightarrow{B}}\,\underrightarrow{\underrightarrow{F}} + \Phi(\underrightarrow{H}, \underrightarrow{\underrightarrow{B}})\big\rangle = \int \big\langle \underrightarrow{\underrightarrow{B}}\,\underrightarrow{\underrightarrow{F}} - \frac{1}{4}\underrightarrow{\underrightarrow{B}}_s\underrightarrow{\underrightarrow{B}}_s\gamma + \underrightarrow{\underrightarrow{B}}_m{*}\underrightarrow{\underrightarrow{B}}_m + \underrightarrow{\underrightarrow{B}}_h{*}\underrightarrow{\underrightarrow{B}}_h\big\rangle$$

$$= \int \big\langle \underrightarrow{\underrightarrow{F}}_s\,\underrightarrow{\underrightarrow{F}}_s\gamma^- + \frac{1}{4}\underrightarrow{\underrightarrow{F}}_m{*}\underrightarrow{\underrightarrow{F}}_m + \frac{1}{4}\underrightarrow{\underrightarrow{F}}_h{*}\underrightarrow{\underrightarrow{F}}_h\big\rangle$$

Gravitational action

$$S_s = \int \left\langle \underset{\rightrightarrows}{B}_s \underset{\rightrightarrows}{F}_s + \Phi_s(\underset{\rightrightarrows}{B}_s) \right\rangle \; = \int \left\langle \underset{\rightrightarrows}{B}_s \tfrac{1}{2}\left(\underset{\rightrightarrows}{R} + \tfrac{1}{8} M^2 \underset{\rightharpoondown}{e}\,\underset{\rightharpoondown}{e}\right) - \tfrac{1}{4} \underset{\rightrightarrows}{B}_s \underset{\rightrightarrows}{B}_s \gamma \right\rangle$$

$$\delta \underset{\rightrightarrows}{B}_s \to \underset{\rightrightarrows}{B}_s = \left(\underset{\rightrightarrows}{R} + \tfrac{1}{8} M^2 \underset{\rightharpoondown}{e}\,\underset{\rightharpoondown}{e}\right)\gamma^- \qquad \text{pseudoscalar:} \;\; \gamma = \gamma_0\gamma_1\gamma_2\gamma_3$$

$$S_s = \tfrac{1}{4} \int \left\langle \left(\underset{\rightrightarrows}{R} + \tfrac{1}{8} M^2 \underset{\rightharpoondown}{e}\,\underset{\rightharpoondown}{e}\right)\left(\underset{\rightrightarrows}{R} + \tfrac{1}{8} M^2 \underset{\rightharpoondown}{e}\,\underset{\rightharpoondown}{e}\right)\gamma^- \right\rangle = \int \left\langle \underset{\rightrightarrows}{F}_s \underset{\rightrightarrows}{F}_s \gamma^- \right\rangle$$

$$\left\langle \underset{\rightrightarrows}{R}\underset{\rightrightarrows}{R}\gamma^- \right\rangle = \underset{\rightharpoondown}{d} \left\langle \left(\underset{\rightharpoondown}{\omega}\,\underset{\rightharpoondown}{d}\,\underset{\rightharpoondown}{\omega} + \tfrac{1}{3}\underset{\rightharpoondown}{\omega}\,\underset{\rightharpoondown}{\omega}\,\underset{\rightharpoondown}{\omega}\right)\gamma^- \right\rangle \qquad \leftarrow \text{Chern-Simons}$$

$$\tfrac{1}{4!}\left\langle \underset{\rightharpoondown}{e}\,\underset{\rightharpoondown}{e}\,\underset{\rightharpoondown}{e}\,\underset{\rightharpoondown}{e}\,\gamma^- \right\rangle = \underline{e} \qquad \leftarrow \text{volume element}$$

$$\left\langle \underset{\rightharpoondown}{e}\,\underset{\rightharpoondown}{e}\,\underset{\rightrightarrows}{R}\,\gamma^- \right\rangle = \underline{e}\,R \qquad \leftarrow \text{curvature scalar}$$

$$S_s = \frac{\Lambda}{12} \int \underline{e}\,(R + 2\Lambda) \qquad \text{cosmological constant:} \quad \Lambda = \tfrac{3}{4} M^2$$

Action for everything

$$\underset{\cdot\underset{\rightharpoondown}{\cdot}}{\underset{\rightharpoondown}{F}} = \underset{\rightharpoondown}{d}\,\underset{\cdot\rightharpoondown}{A} + \underset{\cdot\rightharpoondown}{A}\,\underset{\cdot\rightharpoondown}{A} = \left(\underset{\rightharpoondown}{d}\,\underset{\rightharpoondown}{H} + \underset{\rightharpoondown}{H}\,\underset{\rightharpoondown}{H}\right) + \left(\underset{\rightharpoondown}{d}\,\underset{\rightharpoondown}{G} + \underset{\rightharpoondown}{G}\,\underset{\rightharpoondown}{G}\right) + \left(\underset{\rightharpoondown}{d}\,\underset{\cdot}{\psi} + \underset{\rightharpoondown}{H}\,\underset{\cdot}{\psi} + \underset{\cdot}{\psi}\,\underset{\rightharpoondown}{G}\right)$$

Modified BF action for everything, using $\underset{\rightharpoondown}{\underset{\rightharpoondown}{\dot{B}}} = \underset{\rightharpoondown}{\underset{\rightharpoondown}{B}} + \underline{\dot{B}}$:

$$S = \int \left\langle \underset{\rightharpoondown}{\underset{\rightharpoondown}{\dot{B}}}\,\underset{\cdot\rightharpoondown}{\underset{\cdot\rightharpoondown}{F}} + \underline{\Phi}\left(\underset{\rightharpoondown}{H}, \underset{\rightharpoondown}{G}, \underset{\rightharpoondown}{\underset{\rightharpoondown}{B}}\right)\right\rangle$$

$$= \int \left\langle \underline{\dot{B}}\left(\underset{\rightharpoondown}{d}\,\underset{\cdot}{\psi} + \underset{\rightharpoondown}{H}\,\underset{\cdot}{\psi} + \underset{\cdot}{\psi}\,\underset{\rightharpoondown}{G}\right) + \underset{\rightharpoondown}{\underset{\rightharpoondown}{B}}\,\underset{\rightharpoondown}{\underset{\rightharpoondown}{F}} - \frac{1}{4}\underset{\rightharpoondown}{\underset{\rightharpoondown}{B_s}}\,\underset{\rightharpoondown}{\underset{\rightharpoondown}{B_s}}\gamma + B_{\underset{\rightharpoondown}{\underset{\rightharpoondown}{m,h,G}}} * B_{\underset{\rightharpoondown}{\underset{\rightharpoondown}{m,h,G}}}\right\rangle$$

Fermionic part, using **anti-ghost Grassmann** 3-form, $\underline{\dot{B}} = \underline{e}\,\dot{\psi}\,\vec{e}$:

$$S_f = \int \left\langle \underline{\dot{B}}\left(\underset{\rightharpoondown}{d}\,\underset{\cdot}{\psi} + \underset{\rightharpoondown}{H}\,\underset{\cdot}{\psi} + \underset{\cdot}{\psi}\,\underset{\rightharpoondown}{G}\right)\right\rangle$$

$$= \int \left\langle \underline{e}\,\dot{\psi}\,\vec{e}\left(\underset{\rightharpoondown}{d}\,\underset{\cdot}{\psi} + \frac{1}{2}\underset{\rightharpoondown}{\omega}\,\underset{\cdot}{\psi} + \frac{1}{4}\underset{\rightharpoondown}{e}\,\phi\,\underset{\cdot}{\psi} + \underset{\rightharpoondown}{B}\,\underset{\cdot}{\psi} + \underset{\rightharpoondown}{W}\,\underset{\cdot}{\psi} + \underset{\cdot}{\psi}\,\underset{\rightharpoondown}{G}\right)\right\rangle$$

$$= \int d^4\!\underline{x}\,|e|\left\langle \dot{\psi}\gamma^{\mu}(e_{\mu})^{i}\left(\partial_i\underset{\cdot}{\psi} + \frac{1}{4}\omega_{i}{}^{\mu\nu}\gamma_{\mu\nu}\underset{\cdot}{\psi} + B_i\underset{\cdot}{\psi} + W_i\underset{\cdot}{\psi} - \underset{\cdot}{\psi}G_i\right) + \dot{\psi}\,\phi\,\underset{\cdot}{\psi}\right\rangle$$

Why this Lie algebra

$$\underset{\cdot}{\underrightarrow{A}} = \underrightarrow{H} + \underrightarrow{G} + \underset{\cdot}{\psi} = \begin{bmatrix} \underrightarrow{H}^{+} & \underset{\cdot}{\psi}^{-} \\ & \underrightarrow{G}^{-} \end{bmatrix}$$

$$= \begin{bmatrix}
\frac{1}{2}\underrightarrow{\omega}_L + i\underrightarrow{W}^3 & i\underrightarrow{W}^1 + \underrightarrow{W}^2 & -\frac{1}{4}\underrightarrow{e}_R\phi_0^* & \frac{1}{4}\underrightarrow{e}_R\phi_+ & \underset{\cdot}{\nu}_L & \underset{\cdot}{u}^r_L & \underset{\cdot}{u}^g_L & \underset{\cdot}{u}^b_L \\
i\underrightarrow{W}^1 - \underrightarrow{W}^2 & \frac{1}{2}\underrightarrow{\omega}_L - i\underrightarrow{W}^3 & \frac{1}{4}\underrightarrow{e}_R\phi_+^* & \frac{1}{4}\underrightarrow{e}_R\phi_0 & \underset{\cdot}{e}_L & \underset{\cdot}{d}^r_L & \underset{\cdot}{d}^g_L & \underset{\cdot}{d}^b_L \\
-\frac{1}{4}\underrightarrow{e}_L\phi_0 & \frac{1}{4}\underrightarrow{e}_L\phi_+ & \frac{1}{2}\underrightarrow{\omega}_R + i\underrightarrow{B} & & \underset{\cdot}{\nu}_R & \underset{\cdot}{u}^r_R & \underset{\cdot}{u}^g_R & \underset{\cdot}{u}^b_R \\
\frac{1}{4}\underrightarrow{e}_L\phi_+^* & \frac{1}{4}\underrightarrow{e}_L\phi_0^* & & \frac{1}{2}\underrightarrow{\omega}_R - i\underrightarrow{B} & \underset{\cdot}{e}_R & \underset{\cdot}{d}^r_R & \underset{\cdot}{d}^g_R & \underset{\cdot}{d}^b_R \\
 & & & & i\underrightarrow{B} & & & \\
 & & & & & \frac{-i}{3}\underrightarrow{B} + i\underrightarrow{G}^{3+8} & i\underrightarrow{G}^1 - \underrightarrow{G}^2 & i\underrightarrow{G}^4 - \underrightarrow{G}^5 \\
 & & & & & i\underrightarrow{G}^1 + \underrightarrow{G}^2 & \frac{-i}{3}\underrightarrow{B} - i\underrightarrow{G}^{3+8} & i\underrightarrow{G}^6 - \underrightarrow{G}^7 \\
 & & & & & i\underrightarrow{G}^4 + \underrightarrow{G}^5 & i\underrightarrow{G}^6 + \underrightarrow{G}^7 & \frac{-i}{3}\underrightarrow{B} - \frac{2i}{\sqrt{3}}\underrightarrow{G}^8
\end{bmatrix}$$

Note: Only one generation, and fermion masses not quite right.

For three generations: $\underset{\cdot}{\underrightarrow{A}} \in \underrightarrow{so}(1,7) + \underrightarrow{so}(8) + 3 * \underset{\cdot}{\mathbb{R}}(8 \times 8) = \; ?$

BIG Lie algebra: $n = 28 + 28 + 3 * 64 = 248$

Real simple compact Lie groups

rank	group	a.k.a.	dim	name
r	A_r	$SU(r+1)$	$r(r+2)$	special unitary group
r	B_r	$SO(2r+1)$	$r(2r+1)$	odd special orthogonal group
r	C_r	$Sp(2r)$	$r(2r+1)$	symplectic group
$r>2$	D_r	$SO(2r)$	$r(2r-1)$	even special orthogonal group
2	G_2		14	G2
4	F_4		52	F4
6	E_6		78	E6
7	E_7		133	E7
8	E_8		248	E8

"E8 is perhaps the most beautiful structure in all of mathematics, but it's very complex."

— Hermann Nicolai

Triality decomposition of E8

John Baez in TWF90:

> we now look at the vector space
>
> $$so(8) + so(8) + end(V) + end(S^+) + end(S^-)$$
>
> ...Since $so(8)$ has a representation as linear transformations of V, it has two representations on $end(V)$, corresponding to **left and right matrix multiplication**; glomming these two together we get a representation of $so(8) + so(8)$ on $end(V)$. Similarly we have representations of $so(8) + so(8)$ on $end(S^+)$ and $end(S^-)$. Putting all this stuff together we get a Lie algebra, if we do it right - and it's $E8$.

$$E = H + G + \Psi_I + \Psi_{II} + \Psi_{III} \qquad \in \mathrm{Lie}(E8)$$

$$\begin{array}{lll} [H, \Psi_I] = H\,\Psi_I & [H, \Psi_{II}] = H^+\,\Psi_{II} & [H, \Psi_{III}] = H^-\,\Psi_{III} \\ [G, \Psi_I] = \Psi_I\,G & [G, \Psi_{II}] = \Psi_{II}\,G^+ & [G, \Psi_{III}] = \Psi_{III}\,G^- \end{array}$$

E8 T.O.E.

Build a real form of complex **E8** by using $Cl^2(1,7) = so(1,7)$ instead of $Cl^2(8) = so(8)$. Then **E8 T.O.E. connection** is:

$$\underset{\cdot\rightharpoondown}{A} = \underset{\rightharpoondown}{H} + \underset{\rightharpoondown}{G} + \underset{\cdot}{\Psi}_I + \underset{\cdot}{\Psi}_{II} + \underset{\cdot}{\Psi}_{III} =$$

something like

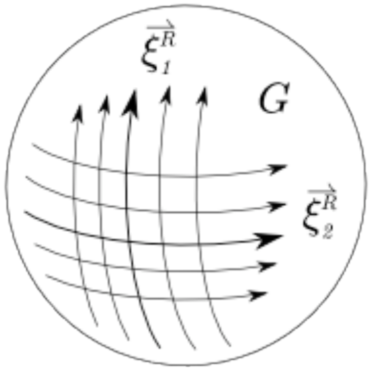
$$\begin{bmatrix} \frac{1}{2}\underset{\rightharpoondown}{\omega}_L + i\underset{\rightharpoondown}{W}^3 & i\underset{\rightharpoondown}{W}^1 + \underset{\rightharpoondown}{W}^2 & -\frac{1}{4}\underset{\rightharpoondown}{e}_R\phi_0^* & \frac{1}{4}\underset{\rightharpoondown}{e}_R\phi_+ \\ i\underset{\rightharpoondown}{W}^1 - \underset{\rightharpoondown}{W}^2 & \frac{1}{2}\underset{\rightharpoondown}{\omega}_L - i\underset{\rightharpoondown}{W}^3 & \frac{1}{4}\underset{\rightharpoondown}{e}_R\phi_+^* & \frac{1}{4}\underset{\rightharpoondown}{e}_R\phi_0 \\ -\frac{1}{4}\underset{\rightharpoondown}{e}_L\phi_0 & \frac{1}{4}\underset{\rightharpoondown}{e}_L\phi_+ & \frac{1}{2}\underset{\rightharpoondown}{\omega}_R + i\underset{\rightharpoondown}{B} & \\ \frac{1}{4}\underset{\rightharpoondown}{e}_L\phi_+^* & \frac{1}{4}\underset{\rightharpoondown}{e}_L\phi_0^* & & \frac{1}{2}\underset{\rightharpoondown}{\omega}_R - i\underset{\rightharpoondown}{B} \end{bmatrix} + \begin{bmatrix} i\underset{\rightharpoondown}{B} & & & \\ & \frac{-i}{3}\underset{\rightharpoondown}{B} + i\underset{\rightharpoondown}{G}^{3+8} & i\underset{\rightharpoondown}{G}^1 - \underset{\rightharpoondown}{G}^2 & i\underset{\rightharpoondown}{G}^4 - \underset{\rightharpoondown}{G}^5 \\ & i\underset{\rightharpoondown}{G}^1 + \underset{\rightharpoondown}{G}^2 & \frac{-i}{3}\underset{\rightharpoondown}{B} - i\underset{\rightharpoondown}{G}^{3+8} & i\underset{\rightharpoondown}{G}^6 - \underset{\rightharpoondown}{G}^7 \\ & i\underset{\rightharpoondown}{G}^4 + \underset{\rightharpoondown}{G}^5 & i\underset{\rightharpoondown}{G}^6 + \underset{\rightharpoondown}{G}^7 & \frac{-i}{3}\underset{\rightharpoondown}{B} - \frac{2i}{\sqrt{3}}\underset{\rightharpoondown}{G}^8 \end{bmatrix}$$

$$+ \begin{bmatrix} \underset{\cdot}{\nu}^e_L & \underset{\cdot}{u}^r_L & \underset{\cdot}{u}^g_L & \underset{\cdot}{u}^b_L \\ \underset{\cdot}{e}_L & \underset{\cdot}{d}^r_L & \underset{\cdot}{d}^g_L & \underset{\cdot}{d}^b_L \\ \underset{\cdot}{\nu}^e_R & \underset{\cdot}{u}^r_R & \underset{\cdot}{u}^g_R & \underset{\cdot}{u}^b_R \\ \underset{\cdot}{e}_R & \underset{\cdot}{d}^r_R & \underset{\cdot}{d}^g_R & \underset{\cdot}{d}^b_R \end{bmatrix} + \begin{bmatrix} \underset{\cdot}{\nu}^\mu_L & \underset{\cdot}{c}^r_L & \underset{\cdot}{c}^g_L & \underset{\cdot}{c}^b_L \\ \underset{\cdot}{\mu}_L & \underset{\cdot}{s}^r_L & \underset{\cdot}{s}^g_L & \underset{\cdot}{s}^b_L \\ \underset{\cdot}{\nu}^\mu_R & \underset{\cdot}{c}^r_R & \underset{\cdot}{c}^g_R & \underset{\cdot}{c}^b_R \\ \underset{\cdot}{\mu}_R & \underset{\cdot}{s}^r_R & \underset{\cdot}{s}^g_R & \underset{\cdot}{s}^b_R \end{bmatrix} + \begin{bmatrix} \underset{\cdot}{\nu}^\tau_L & \underset{\cdot}{t}^r_L & \underset{\cdot}{t}^g_L & \underset{\cdot}{t}^b_L \\ \underset{\cdot}{\tau}_L & \underset{\cdot}{b}^r_L & \underset{\cdot}{b}^g_L & \underset{\cdot}{b}^b_L \\ \underset{\cdot}{\nu}^\tau_R & \underset{\cdot}{t}^r_R & \underset{\cdot}{t}^g_R & \underset{\cdot}{t}^b_R \\ \underset{\cdot}{\tau}_R & \underset{\cdot}{b}^r_R & \underset{\cdot}{b}^g_R & \underset{\cdot}{b}^b_R \end{bmatrix}$$

Geometry of Yang-Mills theory

Start with a **Lie group manifold** (*torsor*), G, coordinatized by y^p.
Two sets of invariant vector fields (*symmetries, **Killing vector** fields*):

$$\vec{\xi}^L_A(y)\, \underset{\to}{d} g = T_A\, g(y) \qquad \vec{\xi}^R_A(y)\, \underset{\to}{d} g = g(y)\, T_A$$



Lie derivative: $[\vec{\xi}^R_A, \vec{\xi}^R_B] = C_{AB}{}^C \vec{\xi}^R_C$

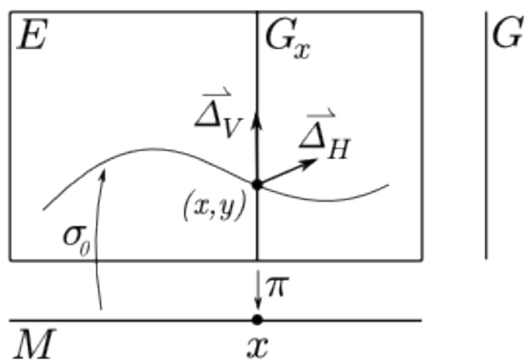
Lie bracket: $[T_A, T_B] = C_{AB}{}^C T_C$

Killing form (***Minkowski metric***): $g_{AB} = C_{AC}{}^D C_{BD}{}^C$

Maurer-Cartan form (***frame***): $\underset{\to}{\mathcal{I}} = \underset{\to}{d} y^p (\xi^R_p)^A T_A$

Entire space of a **principal bundle**: $E \sim M \times G$
Ehresmann principal bundle connection over patches of E:

$$\underset{\to}{\vec{\mathcal{E}}}(x,y) = \underset{\to}{d} x^i A_i{}^B(x)\, \vec{\xi}^L_B(y) + \underset{\to}{d} y^p \vec{\partial}_p$$



Gauge field **connection** over M:

$$\underset{\to}{A}(x) = \sigma_0^* \underset{\to}{\vec{\mathcal{E}}}\, \underset{\to}{\mathcal{I}} = \underset{\to}{d} x^i A_i{}^B(x)\, T_B$$

Cartan subalgebra and charges

Mutually **commuting** set of r **Lie algebra** generators:

$$\{T_1, T_2, ..., T_r\} \qquad\qquad [T_i, T_j] = 0$$

Cartan subalgebra: $C = c^i T_i \quad \in \mathrm{Lie(G)}$

Eigenvalues, α^a, and **eigenvectors**, $V_a \in \mathrm{Lie(G)}$, using the Lie bracket:

$$[C, V_a] = \alpha^a V_a = \sum_i c^i \alpha_i^a V_a$$

Unique eigenvalue for each of the $(n-r)$ eigenvectors, corresponding to $(n-r)$ **roots**, α_i^a, in r dimensional vector space.

Cartan subalgebra of the standard model and gravity:

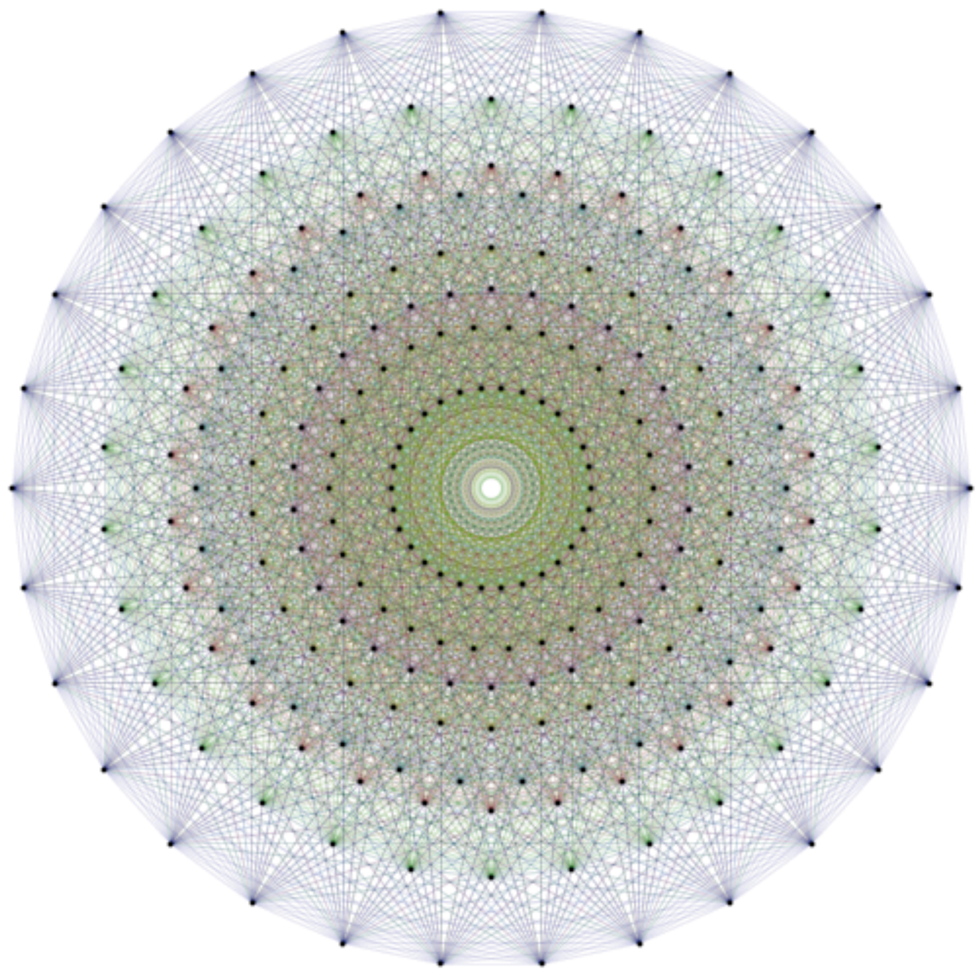
$$C = \tfrac{1}{2}\omega^{01}\gamma_{01} + \tfrac{1}{2}\omega^{12}\gamma_{12} + W^3 i\Sigma_3 + BiY + G^3 i\lambda_3 + G^8 i\lambda_8$$

Eigenvectors are elementary particles, roots are their charges:

$$\alpha(e_L) = (\pm\tfrac{1}{2}, \mp\tfrac{1}{2}, -1, -1, 0, 0)$$

E8 roots

$\mathrm{Lie}(E8)$ has $(248-8)=240$ roots in 8D space — vertices of $P4_{2,1}$:



$E8$ T.O.E.: Each vertex corresponds to an elementary particle.

Reducing E8 to the standard model

One particularly interesting way $e8$ can be broken down:

$$
\begin{aligned}
e8 &= e6 + su(3) + 54 \times 3 \\
&= so(1,9) + u(1) + 32 + su(3) + 54 \times 3 \\
&= so(1,3) + su(2) + su(2) + u(1) + 4 \times 8 + u(1) + 32 + su(3) + 54 \times 3 \\
&\to \tfrac{1}{2}\omega + W + B + \tfrac{1}{4}e\phi + G + 3 \times \psi + X?
\end{aligned}
$$

How does this $e8$ breakdown relate to **e8 triality decomposition**?

$$
\begin{aligned}
e8 &= so(1,7) + so(8) + 3 \times 8 \times 8 \\
&= so(1,3) + so(4) + 4 \times 4 + so(6) + so(2) + 6 \times 2 + 3 \times 8 \times 8 \\
&= so(1,3) + su(2) + su(2) + 4 \times 4 + su(4) + u(1) + 6 \times 2 + 3 \times 8 \times 8
\end{aligned}
$$

Discussion

What is done:

- All **gauge fields**, **gravity**, and Higgs in **one** **connection**, with fermions as **BRST ghosts**.

To do:

- Will particle assignments work with **E8**? (Get the mass matrix?)
- Why is the action what it is? (How's symmetry breaking happen?)
- Is a four dimensional base **manifold** emergent?
- How does this theory get quantized? (LQG methods should apply.)
    - Natural explanation for QM as a bonus?

What this theory will mean, if it all works:

- Gravitational **frame** and Higgs are intimately related.
- Naturally combines standard model with gravity — so it's a **T.O.E.**
    - (It's also a U.F.T., but I don't like to call it that.)
- Our universe is a very pretty shape!

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BRST gauge fixing

$\delta \underline{L} = 0$ under **gauge transformation**: $\delta \underset{\rightharpoondown}{A} = -\underset{\rightharpoondown}{\nabla} C = -\underset{\rightharpoondown}{d} C - [\underset{\rightharpoondown}{A}, C]$

Account for gauge part of $\underset{\rightharpoondown}{A}$ by introducing **Grassmann** valued **ghosts**, $\underset{\cdot}{C} \in \mathrm{L\underset{\cdot}{i}e}(G)_g$, **anti-ghosts**, $\dot{\underline{B}}$, **partners**, $\underline{\lambda}$, and **BRST transformation**:

$$
\begin{aligned}
\underset{\cdot}{\delta} \underset{\rightharpoondown}{A} &= -\underset{\rightharpoondown}{\nabla} \underset{\cdot}{C} & \qquad \underset{\cdot}{\delta} \underset{\cdot}{C} &= -\tfrac{1}{2}[\underset{\cdot}{C}, \underset{\cdot}{C}] \\
\underset{\cdot}{\delta} \underset{\rightharpoondown}{\underset{\rightharpoondown}{B}} &= [\underset{\rightharpoondown}{\underset{\rightharpoondown}{B}}, \underset{\cdot}{C}] & \qquad \underset{\cdot}{\delta} \dot{\underline{B}} &= \underline{\lambda} \\
\underset{\cdot}{\delta} \underline{\lambda} &= 0 & &
\end{aligned}
$$

This satisfies $\underset{\cdot}{\delta} \underline{L} = 0$ and $\underset{\cdot}{\delta}\underset{\cdot}{\delta} = 0$.

Choose a **BRST potential**, $\dot{\underline{\Psi}} = \langle \dot{\underline{B}} \underset{\rightharpoondown}{A} \rangle$, to get new Lagrangian:

$$
\underline{L}' = \underline{L} + \underset{\cdot}{\delta} \dot{\underline{\Psi}} = \underline{L} + \langle \underline{\lambda} \underset{\rightharpoondown}{A}_g \rangle + \langle \dot{\underline{B}} \underset{\rightharpoondown}{\nabla} \underset{\cdot}{C} \rangle
$$

BRST partners act as Lagrange multipliers; **effective Lagrangian**:

$$
\underline{L}^{\mathrm{eff}} = \underline{L}[\underset{\rightharpoondown}{\underset{\rightharpoondown}{B}}', \underset{\rightharpoondown}{A}'] + \langle \dot{\underline{B}} \underset{\rightharpoondown}{\nabla}' \underset{\cdot}{C} \rangle
$$

BRST extended connection

Replace pure gauge part of connection with ghosts:

$$\underset{\cdot\rightarrow}{A} = \underset{\rightarrow}{A}' + \underset{\cdot}{C}$$

BRST extended curvature:

$$\underset{\substack{\cdot\rightarrow\\ \cdot\rightarrow}}{F} = \underset{\rightarrow}{d}\underset{\cdot\rightarrow}{A} + \frac{1}{2}[\underset{\cdot\rightarrow}{A}, \underset{\cdot\rightarrow}{A}] = \underset{\substack{\rightarrow\\ \rightarrow}}{F}' + \underset{\rightarrow}{\nabla}'\underset{\cdot}{C} + \frac{1}{2}[\underset{\cdot}{C}, \underset{\cdot}{C}]$$

$$= \left(\underset{\rightarrow}{d}\underset{\rightarrow}{A}' + \underset{\rightarrow}{A}'\underset{\rightarrow}{A}'\right) + \left(\underset{\rightarrow}{d}\underset{\cdot}{C} + [\underset{\rightarrow}{A}', \underset{\cdot}{C}]\right) + \frac{1}{2}[\underset{\cdot}{C}, \underset{\cdot}{C}]$$

Effective Lagrangian, with $\dot{\underline{B}}' = \underline{B}' + \dot{\underline{B}}$:

$$\underline{L}^{\text{eff}} = \left\langle \dot{\underline{B}}' \underset{\substack{\cdot\rightarrow\\ \cdot\rightarrow}}{F} + \underline{\Phi}(\underset{\rightarrow}{A}', \underline{B}') \right\rangle$$

Crazy idea: **Fermions are gauge ghosts**

$$\underset{\rightarrow}{A}' = \underset{\rightarrow}{H} + \underset{\rightarrow}{G} = \left(\frac{1}{2}\underset{\rightarrow}{\omega} + \frac{1}{4}\underset{\rightarrow}{e}\phi + \underset{\rightarrow}{B} + \underset{\rightarrow}{W}\right) + \underset{\rightarrow}{G}$$

$$\underset{\cdot}{C} = \underset{\cdot}{\psi} = \left(\underset{\cdot}{\nu} + \underset{\cdot}{e} + \underset{\cdot}{u}^{r,b,g} + \underset{\cdot}{d}^{r,b,g}\right)$$