A mathematical conflict between the Special relativity velocity-addition law and its own postulate

Abstract
Absurdities arising from Einstein's velocity-addition law have been discussed since the theory's formulation. Most of these have been dismissed as being philosophical arguments and supporters of Special relativity theory are of the opinion that if the math is not faulted they are ready to live with the paradoxes. Here, we now demonstrate a mathematical contradiction internal to the theory itself. We show that when applied to light there is no way to mathematically reconcile the Einstein velocity-addition law with the second postulate of the theory which may have a fatal consequence.

Introduction
Einstein's second postulate of Special relativity and the accompanying velocity-addition proposal was an attempt to explain the result of an optical experiment which had been made to measure the velocity of the earth through space. The result seemed to suggest that if there was no earth-bound matter medium through which light propagated, then the relative (i.e. resultant) velocity of light was always $c$ irrespective of the motion of the receiver's motion at velocity $v$. Thus no matter at what velocity two travelling entities A and B approach each other at velocities $V_A$ and $V_B$, their resultant velocity, $V_{AB}$ can never be greater than $c$. Stated mathematically,

$$V_{AB} = \frac{V_A + V_B}{1 + \frac{V_AV_B}{c^2}} \tag{1}$$

This is different from the Galilean velocity-addition law

$$V_{AB} = V_A + V_B \tag{2}$$

At the small velocities available for testing in the laboratory, the differences between the two are very tiny given the equations. At large velocities that can be used to discuss the two velocity-addition laws to bring out contradictions or absurdities, various alibis can be found to explain any theoretical inconsistencies. Notable among these are that verification would be impossible both in theory and in practice because it will be impossible for material objects to travel at velocities approaching light speed due to the objects shrinking due to length contraction which would make such objects disappear as light speed is approached, and because of the hypothetical increasing mass with velocity it would become more and more difficult to reach light speed. These alibis have made it difficult to interrogate the theoretical predictions.

In this contribution, we first explain what relative velocity means and then demonstrate how it is used in calculations where velocity addition is required between moving objects in the Galilean way and the Special relativity way. We then apply the velocity-addition law to light itself as
dictated by the Galilean and Special relativity theories before discussing the implications of the differing results.

**What does relative velocity mean and how is it used in physics?**

Consider two cars on a linear race-track 2000 metres apart.

If Car A (green) races at 3m/s towards O and Car B (red) does the same at 3m/s, by using the velocity-addition law, we can find the Time of collision with the formula

\[
\text{Time of collision} = \frac{\text{Distance, } D}{\text{relative velocity of A and B (} V_{AB}\text{)}}
\]

Using Galilean velocity-addition law,

\[
V_{AB} = V_A + V_B = 6 \text{m/s}
\]

Therefore, Time of collision = \(\frac{2000}{6}\) = approx. 333.333 seconds.

We can also use two different velocities such as Car A travelling at 2m/s and Car B at 4m/s. Time of collision will be same but the distance covered by each car will be different, i.e. individual car's velocity multiplied by the time of collision (333.333 seconds).

Using Einstein velocity-addition law,

\[
V_{AB} = \frac{V_A + V_B}{1 + \frac{V_A V_B}{c^2}} = \frac{6}{1 + (9/ 9 \times 10^{-16})} \text{ m/s}
\]

assuming \(c = 3 \times 10^8 \text{m/s}\). Therefore,

\[
\text{Time of collision} = \frac{\text{Distance, } D}{\text{relative velocity of A and B (} V_{AB}\text{)}} = \text{approx. 333.333 (+ 10^{-13}) seconds}
\]
Both velocity-addition laws yield results indistinguishable at commonly occurring speeds and short distances. There are hypothetical examples in the literature using spacecraft flying at higher speeds towards each other with a sum of velocities higher than \( c \), with more obvious differences between both velocity-addition laws, with the Galilean supporting a relative velocity value higher than \( c \) and the Special relativity velocity-addition showing a curiously lower than \( c \) relative velocity value despite the sum of \( V_A \) and \( V_B \) being higher than \( c \). For example, two spacecrafts travelling towards each other each with a velocity 0.9\( c \), have a relative velocity 1.8\( c \) with Galilean velocity-addition and 0.99\( c \) with the Einstein version. This curiosity is explained away by saying that since no material object can actually travel at speeds approaching \( c \) according to the theory no absurdity would arise in such a scenario.

Since, it is known that light waves can unarguably travel at \( c \), the above alibis are not available and so we use them to test the internal mathematical consistency of both velocity-addition laws. Here we use the particle description of light as photons only for convenience.

**Application of the velocity-addition law to light itself**

Now let us race two photons, emitted simultaneously at \( O \) in opposite directions, and sent to mirrors one light-year away and reflected back to \( O \) for detection. The contraption is to avoid alibi and disputes about synchronization and simultaneity of emission and reception at spatially separated events which are sometimes brought up to explain observed contradictions.

![Diagram of photon race](attachment:image.png)

\[
\begin{align*}
\text{M}_1 & \quad \text{M}_2 \\
\text{1 light-year} & \quad \text{1 light-year} \\
D = 2 \text{ light-years} \\
\end{align*}
\]

[*A light-year is the distance travelled by light in one year, (i.e. \( c \times \text{one year duration} \)]

Figure 2

Photon A has \( V_A = c_A = c = 299,792,458 \text{m/s} \) (approx. \( 3 \times 10^8 \text{m/s} \))

Photon B has \( V_B = c_B = c = 299,792,458 \text{m/s} \) (approx. \( 3 \times 10^8 \text{m/s} \))

Each photon reaches its respective mirror after one year duration and is reflected back.

For Galilean velocity-addition law, after one year, the two photons A and B move towards each other with
Relative velocity of Photon A and Photon B, $V_{AB} = V_A + V_B$

$$= c_A + c_B$$

$$= 2c$$

Time of collision = \( \frac{\text{Distance, D}}{\text{relative velocity of Photon A and Photon B (V}_{AB})} \)

$$= \frac{2 \text{ light-years}}{2c}$$

$$= 1 \text{ year}$$

Adding the one year to reach the mirrors after emission, total time that elapses after emission at O before recombination = 2 years.

For Einstein velocity-addition law, after one year, the two photons A and B are reflected from the mirror and move towards each other with

Relative velocity of Photon A and Photon B = $V_{AB} =\frac{c_A + c_B}{1 + \frac{c_A \cdot c_B}{c^2}}$

$$= \frac{2c}{2}$$

$$= c$$

Thus, no relative velocity can mathematically be greater than $c$ and this is consistent with the Special relativity theory.

Time of collision = \( \frac{\text{Distance, D}}{\text{relative velocity of Photon A and Photon B (V}_{AB})} \)

Time of collision = \( \frac{2 \text{ light-years}}{c} \)

$$= 2 \text{ years}$$

Adding the one year to reach the mirrors after emission, total time that elapses after emission at O before recombination = 3 years.

This difference in timing is enough to discuss both velocity-addition laws for theoretical and mathematical consistency.
Implication of the differing results
Let us now examine the consistency or not in the Einstein velocity-addition law which forms the backbone of Special relativity.

**Implication 1.** For Photons A and B to take two years after leaving the mirror before colliding at O, a distance 1 light-year away from the mirror implies that Photon A for example must have travelled the 1 light-year distance at a velocity, \( c_A = 0.5c \), instead of \( c \). **This will violate the postulate that light speed must always be \( c \).**

**Implication 2.** On the other hand, if Photon A travels the distance at velocity, \( c \) after leaving the mirror it will reach O and collide with Photon B in 1 year. **This will not be in agreement with calculations using the Einstein velocity-addition law which dictate that no relative velocity between entities moving towards each other can exceed \( c \).**

From the fore-going, something must give if BOTH the postulate that light always propagate at a constant \( c \) and the velocity-addition law that relative velocities cannot sum up higher than \( c \) must hold. Proposed solutions to salvage the situation include that (i) time can be dilated and (ii) length contracted to reconcile the Special relativity postulate with the velocity-addition law. To illustrate these proposed solutions:

(i) After leaving the mirror, although both Photons A and B travel at \( c \) in obedience to the second postulate and at a relative velocity to each other also not above \( c \), according to the velocity-addition law, their time of flight is dilated from one year to two years. This way the postulate of light always travelling at \( c \) is obeyed and the velocity-addition law prediction of two years transit from mirror before collision at O is not contravened. This is the time-dilation mechanism at work.

To obtain a time-dilation effect that can be double, i.e. that can double one year (\( t \) flight to two years transit time (\( t' \)), we have the 'time-dilation equation':

\[
t' = t\sqrt{1 - v^2/c^2}
\]  

(4)

From this equation, we can see that no matter how the plus and minus signs are manipulated so that there is a factor of 2 to double transit times, ALL possible values for the velocity, \( v \) that can be obtained are higher than \( c \). This violates the Special relativity postulate that photons cannot propagate faster than \( c \) and the velocity-addition law that relative velocity cannot sum up higher than \( c \).

If we use \( c_{ACB} \) as \( v^2 \) in the equation, since \( c_A = c_B = c \) there is no time dilation at all and so this too cannot be resorted to in order to explain the discrepancy of two Photons A and B colliding over a two light-year distance in a duration of two years.

Indeed, it has been pointed out that if the equation is correct, it would imply that time does not flow for a photon, which leads to the question: if time of emission of a photon is the same as the time of its absorption, how then can photon exist? This has been termed the "photon existence paradox".
In summary, the time-dilation formula fails to reconcile the second postulate of Special relativity with the Einstein velocity-addition law.

(ii) The velocity-addition law holds that Photons A and B must have a relative velocity of \( c \). If in spite of this both Photons A and B must collide after one year over the two light-year distance, in order to avoid conflict with their propagation velocity which will still be \( c \) according to the second postulate, the two light-year distance (\( L \)) must be contracted to a distance of one light-year (\( L' \)). This is the so-called FitzGerald-Lorentz contraction. The formula for this is given by:

\[
L' = L \sqrt{1 - \frac{v^2}{c^2}}
\]

That is,

\[
L' = L \sqrt{1 - \frac{c_A c_B}{c^2}}
\]

In order for the path length traveled to contract by half, i.e. for \( L' \) to become 0.5L, \( v^2 (c_A c_B) \) must equal to 0.75\( c^2 \), and if that is the case, then \( c_A \) and \( c_B \) must each equal to 0.866\( c \). This violates the SR postulate that photons must always travel at \( c \).

On the other hand, if we use \( c_A = c_B = c \) the contraction of the path length (if any), still does not reconcile the postulate of propagation velocity being always \( c \) with the velocity-addition law that relative velocities not sum up higher than \( c \).

**Implication 3.** It is crucial to the mathematical correctness of Special relativity theory that both the Einstein velocity-addition law and the second postulate from which it is deduced must work together. The above analysis shows that there is no mathematical way to reconcile both.

**Concluding remarks**

Can we then entrust our physics to these manipulations of time and distance to preserve the second postulate of Special relativity and the velocity-addition law when in spite of the manipulations, both can still not be consistently reconciled with each other?

Although experimental and philosophical reasons suggesting that a departure from the Galilean velocity-addition law to the Einstein version were not fully justified have been made in the past, this is also now demonstrated on mathematical grounds. A favored proposal of the author is that the latter discovery of galactically abundant dark matter represents a matter medium that can be earth-bound and therefore provide a Galilean solution to the relativity dilemma.