A problem for geometry

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I begin this section with an example of the inconsistencies that arise in mathematics, then consider Riemann's view as it relates to the kind of problem presented. Consider Figure 1, showing two line segments on a Cartesian plane. Let each line segment have unit length, so that the total length of the two line segments is 2 units.

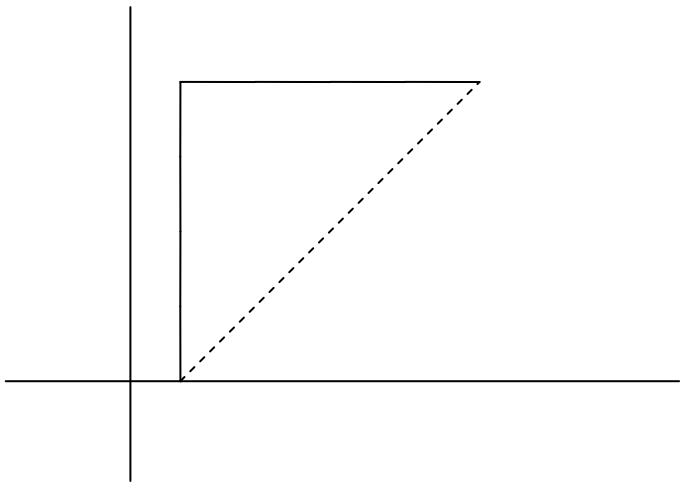

Figure 1 - Triangle

Bisecting each line segment and rearrangement of the parts results in the following configuration (Figure 2).

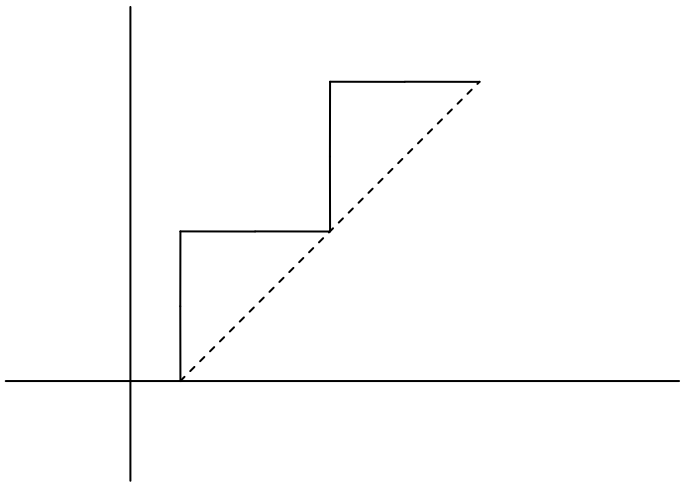

Figure 2 - Two triangles

Rearrangement cannot affect the total length, which remains 2 units. Observe that the number of points at which the shape touches the diagonal increases from 2 to 3 points. Now observe Figure 3.

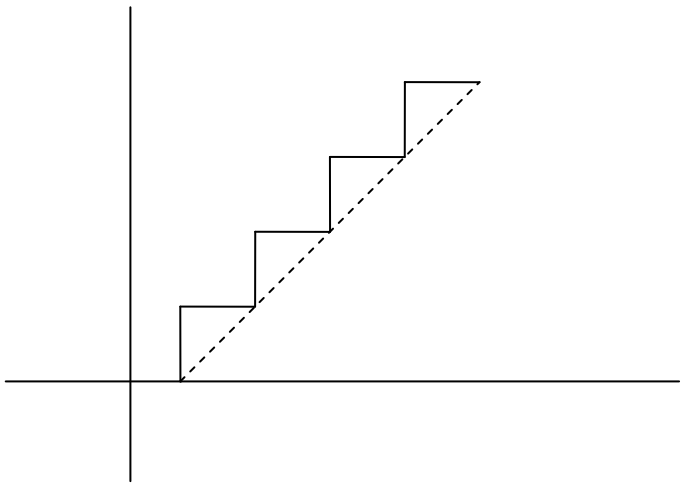

Figure 3 - Four triangles

By symmetry with the previous construction, the total length remains two. and the number of points at which the form touches the diagonal is $1 + 2^n$ where n is the number of bisection and rearrangements of the sub-forms. Again by symmetry, the width of the sub-form is halved with each iteration, and without the aid of a microscope, it soon becomes difficult to distinguish the

difference between this form and that of the diagonal that spans between the end points of the construction.

The symmetry of the system implies that there are triangles 'all the way down', independent of any iterative aspects. The construction converges on a line, and, in the limit, the number of places at which the form touches the diagonal is likewise $1 + 2^{\aleph_0}$, or one plus two to the power of infinity. According to Cantor this covers the diagonal (with a point left over). One might note that this varies from Cantor's determination of $2^{\aleph_0}$ as the cardinality of the reals, but this is unimportant here. Had I chosen to break the triangle up into any multiple of prime numbers of sub triangles, the same reasoning would imply that there is upon the line $3^{\aleph_0}$, $5^{\aleph_0}$, $7^{\aleph_0}$, $11^{\aleph_0}$ …up into higher cardinalities of points on the diagonal. Also, at the limit, one must accept, under analysis, that to each case it is possible to add the apex of each triangle, which increases the number by $\aleph_0$. That any one of these options covers the diagonal is not a paradox. That they all fill the same continuum is a paradox, and indeed inconsistent, for none of these points can land on the same place except at the endpoints of the line because they are based on prime numbers. The matter has nothing to do with their countability, only that *all* describe the same continuum, and they all cover the same diagonal. But they can't all have the same cardinality.

If this reasoning is correct, then Cantor's continuum problem (Gödel 1983):

> How many points are there on a straight line in Euclidean space?

is problematic (Euclidean space is a flat space. Simplistically, it has no spacewarps, and a straight line is properly straight, not the limit of a nearly straight curved line as in Riemannian space).

Separately one may argue that there is a one to one relationship between each point, or line segment on the straight line, and that of the hypotenuse. In itself this is not immediately a problem until it is recognized that while the length of the diagonal is $\sqrt{2}$, by Pythagoras's theorem, the length of the vertical and horizontal line segments remains 2. To a person inspecting the line, or conceiving the line, without prior knowledge of the internal properties of the construct, they would be in no position to distinguish between the given construction and a straight line. In fact it is questionable as to how the construction avoids further inconsistency, given that there seems to be no basis for introducing scale in the limit case.

Because every right-angled triangle may be similarly rearranged, there are an infinite number of examples of this kind in geometry. Because the same may be done with semicircles and every

shape between the semicircle and the triangle, indicates that there is a transfinite number of problems.

We might propose that this implies that at some level, the notion of scale loses sense and that a more holistic theory would accommodate the dichotomy. But this prompts the question: at what length scale does the change take place and, given the symmetry of the first triangle with any triangle at a lower level, what could call a halt to the devolution? More particularly, what kind of principle could act selectively on a particular small triangle and not the one above or below it, and say 'this is the smallest triangle', which is exactly what we find evidence for in the world that we experience – minimum increments in the form of the Planck length and Planck time (Isham 1989).

Riemann (n.d.) refers to similar aspects, stating

> It is known that geometry assumes, as things given, both the notion of space and the first principles of constructions in space. She gives definitions of them which are merely nominal, while the true determinations appear in the form of axioms. The relation of these assumptions remains consequently in darkness; we neither perceive whether and how far their connection is necessary, nor *a priori*, whether it is possible.
>
> From Euclid to Legendre (to name the most famous of modern reforming geometers) this darkness was cleared up neither by mathematicians nor by such philosophers as concerned themselves with it.

He later finds (p.149)

> [I]t is therefore quite likely that the metric relations of space in the infinitely small do not agree with the assumptions of geometry, and in fact one would have to accept [an alternative theorem] as soon as the phenomena can thereby be explained in a simpler way.

No further progress in this matter has been achieved in more recent times. In Hilbert's 'Foundations of Geometry' (1971) the concepts of point, line, plane and the relation of betweenness remain simple (Goheen in Hilbert 1971).

In algebra the matter is similarly confounding. Consider the following (Polster 2004)

Proposition: $1 = 0.999\ldots$

Let	x	=	0.999…
$\Rightarrow$	$10x$	=	9.999…
$\Rightarrow$	$10x - x$	=	9.000...
$\Rightarrow$	$9x$	=	9.000…
$\Rightarrow$	x	=	1.000…

In this kind of solution we recognise the limits of methods of proof, but this is unsatisfying. Intuitively 0.999… and 1.000… represents different real numbers.

One might argue that if we think carefully in limit terms, they should come out to be the same, as the 'proof' given above suggests. Consider the sequence 0.9, 0.99, 0.999, … On the given, any epsilon greater than zero, we can always find an N such that the N minus the term in this sequence differs from 1 by less than epsilon. So the limit of this sequence is equal to 1. But this assumes the epsilon-delta definition of a limit, and that is hazy at best. I don't want to argue against this directly, for the matter becomes more interesting if I accept this argument.

Alternatively one might point out that 0.999… = 9 x 0.111… = 9 x 1/9 = 1. But there is a leap between the second and third expressions, for the idea that 1/9 = 0.111… itself relies on limits. Many readers will have convinced themselves now that I am wrong, or at least that this is not a paradox. Then let us for a moment accept the proof that 0.999… is actually equal to 1.

But now recognise that it is a simple matter to connect this problem to the triangle problem above. Firstly, assign to the perpendicular height of the original triangle from the diagonal a value of 1 unit (this is a different scale to the other units used for the purpose of simplicity of argument). The height of every subsequent triangle is half that height and the end digit is always 5. Then for every reduction in height there is a value that lies between 1 - 0.9, 1- 0.99, … with a value that ends in 5. At the limit there is always a point between 1 – 0.999… and 0 with an end digit of 5. The continuum is never covered by limits.

Strictly speaking the language I have used applies to functions rather than limit sequences and series, but the same reasoning applies in this case. I am not challenging the epsilon delta

definition of limit here. I am simply using the idea to show that one cannot have all of these same constructions to be true in the same system.

If this does not satisfy one, then consider that if 0.999… = 1, then, if I divide the system into sets of ten triangles of total length 2, as opposed to the two triangles shown above, the triangles still go all the way down, as they can only do in a continuum. Then at the limit, the apex lies on the diagonal. That is, at the limit, the distance from the apex of the triangle to the top of the limit is 0.999… and the distance to the base of the triangle is 1.000… Because these are equal (by the second proof) the system can only be a straight line. Then the infinite sum of the line segments, which can only be the same as the original construct, namely 2 units, is equal to the length of the diagonal, √2. The system is inconsistent. Contemporary mathematics is inconsistent. Ouch.