Quaternions and gauge theory
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This note illustrates the origin of gauge theory from quaternions. This is analogous to the Cauchy-Riemann conditions for complex variables. Gauge fields then emerge from the spinor structure of space-time.

The complex variable case is relatively easy to see. A complex number is $z = x + iy$. We then consider a function on this variable as $u + iv = f(z)$. The differential $\frac{\partial f(z)}{\partial z}$ is then

$$\frac{\partial f(z)}{\partial z} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial y}.$$

The derivative is the same if we set $y = 0$ and differentiate with $x$ and if we set $x = 0$ and differentiate with $y$. This is because the differentiation is defined on an infinitesimal disk around a point and we can take either limit. This leads to

$$\frac{\partial f(z)}{\partial z} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

This leads to the well known Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = - \frac{\partial u}{\partial y}.$$

These conditions determine the nature of analytic functions.

We now consider quaternions. The quaternion is an object

$$q = x_0 + ix_1 + jx_2 + kx_3,$$

with the conditions $ij = jk = ki = ijk = -1$ and for any two quaternions the product rule is $ij = -ji$ and generally antisymmetric. We then can look at a similar function $p = f(q) = y_0 + iy_1 + jy_2 + ky_3$. It is then not hard to define the derivative $\frac{\partial f(q)}{\partial q}$ as

$$\frac{\partial f(q)}{\partial q} = \frac{\partial f(q)}{\partial \bar{q}q} \frac{\partial \bar{q}q}{\partial q}$$

$$= \frac{\partial f(q)}{\partial \bar{q}q} (x_0 - i\bar{x}_1 - j\bar{x}_2 - k\bar{x}_3).$$

We use the product rule to find this is

$$\frac{\partial f(q)}{\partial q} = \frac{\partial y_0}{\partial x_0} - \frac{\partial y_1}{\partial x_1}i \wedge i - \frac{\partial y_2}{\partial x_2}j \wedge j - \frac{\partial y_3}{\partial x_3}k \wedge k$$

$$\quad - \left( \frac{\partial y_0}{\partial x_1}i + \frac{\partial y_0}{\partial x_2}j + \frac{\partial y_0}{\partial x_3}k \right) + \left( i \frac{\partial y_1}{\partial |q|^2} + j \frac{\partial y_2}{\partial |q|^2} + k \frac{\partial y_3}{\partial |q|^2} \right) \wedge (-i\bar{x}_1 - j\bar{x}_2 - k\bar{x}_3).$$

This equation is a form of the Clifford algebra product. The wedge $\wedge$ symbol means compute the commutator of the quaternion basis elements. The evaluation of these commutators in the first line gives us zero. This means that as with the case of the Cauchy-Riemann formula that we demand $\frac{\partial y_0}{\partial x_0} = 0$.

The second line evaluates the commutators and reduces the derivatives to

$$\frac{\partial f(q)}{\partial q} = - \left( \frac{\partial y_0}{\partial x_1}i + \frac{\partial y_0}{\partial x_2}j + \frac{\partial y_0}{\partial x_3}k \right).$$
\[
+ \left( \frac{\partial y_3}{x_2} - \frac{\partial y_2}{\partial x_3} \right) i + \left( \frac{\partial y_1}{x_3} - \frac{\partial y_3}{\partial x_1} \right) j + \left( \frac{\partial y_2}{x_1} - \frac{\partial y_1}{\partial x_2} \right) k.
\]

The other portion of the analogue of the Cauchy-Riemann equation is that \( \frac{\partial y_3}{x_2} = \frac{\partial y_2}{\partial x_3} \), in the first component in \( i \), and this holds for the \( j, k \) components as well.

Now that we have these equations we may step back and look at the parts. The top line has an equation that is identical in form to \( \vec{E} = \nabla \vec{A} - \frac{\partial \phi}{\partial t} \). This is identical in form to the electric field.

The second line contains three elements that are components of \( \vec{B} = -\nabla \times \vec{A} \), which are identical in form to magnetic fields. The electric fields are composed of products of 0 components with 1, 2, 3 components and these form the 0, i components of the matrix. The spinor valued equations are computed from commutators of spinor elements, or \( i \) components are from \( k \wedge j \) and form the 3, 1 elements. This gives rise to the electromagnetic tensor components in matrix forms as:

\[
\begin{pmatrix}
0 & \frac{\partial y_0}{\partial x_1} & \frac{\partial y_0}{\partial x_2} & \frac{\partial y_0}{\partial x_3} \\
-\frac{\partial y_0}{\partial x_1} & 0 & -\frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} \\
-\frac{\partial y_0}{\partial x_2} & -\frac{\partial y_1}{\partial x_1} & -\frac{\partial y_2}{\partial x_1} & \frac{\partial y_3}{\partial x_1} \\
-\frac{\partial y_0}{\partial x_3} & -\frac{\partial y_1}{\partial x_3} & -\frac{\partial y_2}{\partial x_3} & -\frac{\partial y_3}{\partial x_3}
\end{pmatrix}
\]

The identification with magnetic fields leads to the matrix in its standard form:

\[
\begin{pmatrix}
0 & -E_1 & -E_2 & -E_3 \\
E_1 & 0 & -B_3 & B_2 \\
E_2 & B_3 & 0 & -B_1 \\
E_3 & -B_2 & B_1 & 0
\end{pmatrix}
\]

The quaternion representation of spacetime demands the occurrence of a structure like gauge fields. It does not predict what that gauge theory is. In what is derived here the theory is \( U(1) \) electromagnetism. This could be derived with gauge covariant operators in a nonabelian setting to get \( SU(2) \) and \( SU(3) \) for weak and QCD interactions in the standard model. The quaternion theory will not be able to address this. For that one must go to octonions, or the \( E_8 \) theory.