The Physical Limitations on Mathematical Abstraction, the Representational Effect of Mathematics on Physical Explanation, and the Resulting Expansion of Computability

Steven P. Sax

The relationship between mathematical representation and physical explanation is discussed. The limitations physics places on mathematical abstraction is exemplified, as well as the effect that changing mathematical representation has on physical understanding. Computation is utilized conceptually to further illustrate these connections. The limits of computation are potentially expanded regarding the role of a self referential state, and physical insights are gleaned from this.

One of the first topics in any course on electrodynamics is the proof for Gauss’s Theorem, which relates the flux of a field vector through a closed surface. Most fascinating about this is how mathematics – relationships and properties of abstract objects according to rules or patterns - explained a spectacular physical theory involving electric fields. But physical theories are never “proven” per se; we simply gather more experimental evidence to support confidence that such a theory works. If further evidence comes along to sufficiently contradict a physical theory, that theory must be modified to accommodate the evidence or abandoned altogether. Mathematical theories by contrast are proven (see (1) for a proof of Gauss’s Theorem). How did Gauss so subtly circumvent all the experimentation by merely moving around abstract symbols on a paper, albeit even if those abstract symbols follow specific rules? The answer is, of course he didn’t. We can’t circumvent the necessary experimentation needed to build confidence in a physical explanation, and in order to use the mathematical proof of Gauss’s Theorem to explain electrical effects, one needs to make certain assumptions as to what the symbols in the theorem represent physically (in addition to the properties of the abstract objects for which the symbols stand). Looking at electric charge as point distributions and expanding the idea of electric force into a vector field, involve major assumptions about the physical nature of electric phenomena – assumptions based on discovery and verification.

But there’s the other issue alluded to above: mathematics also makes assumptions intrinsic to the abstract objects themselves. Unlike the assumptions made between the relationships of physical objects, which can be tested in the physical world, the assumptions of the relationships between abstract objects in mathematics are independent of any physical experimentation. Mathematics tries to reduce these assumptions to very fundamental postulates - so mathematics then comprises the most necessary assumptions which are considered to be absolutely certain; it is upon these “absolutely certain” postulates that the
The rest of mathematics is inferred, or proven. But even those postulates rely on some type of intuition, which ultimately must come to terms with our knowledge of physical reality if they are to represent it. For example, Euclid’s fifth postulate leads to the conclusion that the sum of the angles of a triangle must be 180°, but this is not true over surfaces that curve in space such as a saddle or a sphere (see Figure 1). Einstein’s general theory of relativity gives an even more interesting revelation: the sum of angles of a triangle depends on the gravitational field within the triangle! This is so because gravitational fields curve the space, thus changing what angles are necessary to make a triangle in that space.

Figure 1

The sum of the angles of a triangle can be measured over curved surfaces and confirmed to not add up 180°. But didn’t we just say that mathematical assumptions by definition aren’t put to experiments for validation? Yet here we just tested Euclid’s fifth postulate against a physical observation (and it failed). What happened is that our explanation of physical reality changed based on new physical assumptions. We had to explain curvature resulting from a gravitational field for example, and the assumed rules of physical relationships in curved space can’t be represented by the defined abstract relationships in Euclidean geometry. Our understanding of the physics of flat space (i.e. and without gravity) could still be represented by Euclidean geometry. However, per Einstein’s Theory of Special Relativity, space and time are really part of the same entity – thus spacetime is the more developed explanation of reality. Flat spacetime also would not be represented by Euclidean geometry, but by a pseudo-Euclidean geometry differing mostly in that it is not a metric space. It instead has the interval $T^2 - X^2$, where $T$ is the temporal difference, and $X$ the spatial difference, between two events.

A key then to using mathematics in understanding physical reality is to find good representations which help us explain physics as easily and efficiently as possible. To illustrate this, consider the use of coordinate systems as a way to represent motion. Let’s again discuss the surface of a sphere. Spherical coordinates have abstract properties built in which operate
to easily represent addition of angles.
To simply describe the azimuthal velocity along the sphere, one need only write \( d\phi/dt \). (see \( \phi \) in Figure 2). In Cartesian coordinates, this same motion would have to be represented by

\[
\dot{\phi} = \frac{\rho \dot{z} - z \dot{\rho}}{\rho^2 \sqrt{1 - \left(\frac{z}{\rho}\right)^2}}
\]

where \( \rho = \sqrt{x^2 + y^2 + z^2} \). Perhaps even more revealing is to look at the other way around – motion along a straight line in Cartesian coordinates, represented simply by \( dx/dt \), would be

\[
\dot{x} = \dot{\rho} \cos \phi \cos \theta - \rho \dot{\phi} \sin \phi \cos \theta - \rho \dot{\theta} \cos \phi \sin \theta
\]

where \( \rho = r \), using spherical coordinates. From the perspective of an intelligent being whose intuition is guided in the rules of spherical coordinates, it would seem as though some force were acting to keep an object moving the way it is, and may explain it with various physical assumptions of nature. We however would say it’s moving without any external forces at all and requires only inertia. But look at the last part of that reasoning: it requires only inertia. Any mathematical representation still demands physical assumptions, and changing the mathematical representation actually changes the physical explanation we use. One of the ideas that inspired Einstein in deriving his theories was that free fall is the natural way objects move when there is no external force acting on them, instead of saying it’s due to the force of gravity. Curvature of spacetime reconciled this with explanations already existent in special relativity and Newtonian mechanics. Changing the physical assumption of gravity went hand in hand with changing the mathematical representation of spacetime: gravity became geometry, forming the bedrock of general relativity.

Once physical assumptions are found to have a sufficient confidence level, and a suitable mathematical representation appears to be found, deeper physical understanding can be found by tweaking some of these physical assumptions and/or the mathematical representations, to see what the remaining assumptions would yield. Sometimes physical assumptions are held with such confidence they’re considered laws - like the conservation of momentum. Most amazing is when mathematical representation alone is tweaked to maintain these laws – and seeing what physical explanations must then arise. A good example of this spawned the very foundation of quantum mechanics. Modeling blackbody radiation via the standing wave modes of oscillation of the electromagnetic field, the Rayleigh-Jeans law

\[
I = \frac{2\pi c k T}{\lambda^4}
\]

describes intensity per unit wavelength as: \( I \) is intensity, \( T \) is temperature, \( c \) is the speed of light, and \( k \) is Boltzmann's constant. Dubbed the ultraviolet catastrophe (2), this explanation not only deviated extensively from observation in the lower wavelength spectrum, but violated energy conservation laws. See Figure 3.
To solve this, Planck used the same physical assumptions as Rayleigh-Jean, but substituted a discrete representation for the range of energies, which enabled a theoretical expression for the wavelength intensity distribution

\[ P_\lambda = \frac{2\pi h c^2}{\lambda^5 (e^{(h\nu/kT)} - 1)} \]

(where \( h \) is a new constant) that agreed with experimental data curves of Figure 3. The exact historical motivation for Planck substituting discrete values has been debated (3), but the important point is that he did not have a physical explanation for it. When he presented his theory, most scientists (including Planck!) didn’t consider this quantum concept to be realistic but believed it to be just a mathematical trick. Five years later Einstein rederived Planck’s results by changing the physical assumptions of the cavity oscillations of the electromagnetic field. He proposed they were quantized themselves, and thus light and all electromagnetic radiation were quantized. This led to the concept of photons and photoelectric effect. But that jump of fitting the physics in later was made possible because of the change in mathematical representation. Regardless of how Planck considered the math, we can analyse it from a perspective on infinity. Originally it was assumed the range of energy values should be represented by a continuum of numbers. The infinite possibilities of a continuum form what Cantor would call an uncountable set. But a discrete (yet infinite) set can be shown to have the same number of items as the set of natural numbers via correspondence mapping (formally, “bijection”) (4), and is countable. It’s a different type of infinity, and Cantor described this by denoting each type of infinity by a different ‘cardinality.’ The ability to distinguish between different types of infinity is fascinating, and this insight when applied to abstract representations demanded the physical explanation must change. Photons, quantum mechanics, and the basis of modern technology arose by shifting the terrains of infinity itself. We see now that the two pillars of modern physics - general relativity and quantum physics – owe their discovery in part to purely mathematical changes in representation.
One may ask at this point if we even need this dichotomy between absolute defined assumptions in mathematics, and verified assumptions in physics? After seeing what changing the Cantor cardinality representation does for light, and how changing the intuitive assumptions of coordinate systems can imply or remove forces, maybe physics just is the math? Furthermore, examples exist which appear to bridge purely mathematical concepts to physical phenomena – such as the zeroes of Riemann’s zeta function to quantum energy levels of chaotic systems (5). Perhaps mathematics existentially takes on its own life, and rather than find a mathematical representation to match a physical explanation, one could develop any abstract representation with consistent rules and go seek a physical reality to play it out.

A pragmatic bridge to explore this connection is computation, because it manipulates a representation according to a set of rules - while being limited by the physics of the system making the computation. The limit of computability thus marks the ultimate interface of mathematics and physics. Digital computing is based on symbolic representation. The Church-Turing-Deutsch thesis states 'Every finitely realizable physical system can be perfectly simulated by a universal model computing machine operating by finite means.' Turing hoped his abstracted paper tape model of computation (6) could form a purely abstract theory of computation devoid of physical consequences. But that turns out to be incorrect: the paper is limited in the density of symbols that can be squeezed into it, as is the rate at which the paper can move through the reader. Indeed the physical limit of information content itself for a geometric area is specified through the Holographic Principle (7). From the mathematical perspective, it has been traditionally believed that Godel’s Incompleteness Theorem limits computability (8). This can be expressed through the Halting Problem – the undecidability in determining from an arbitrary computer program and an input, whether the program will finish running or run forever (see Appendix 1) - and the analogous undecidability of self referential statements. Computation reduces ultimately to a sequence of states and thus causality is a key requirement; the undecidability of a self referential statement would indeed then be a limit of computation because it would produce endless loops and destroy the causality. The typical example of a self-referential statement is the liar paradox “this statement is false.” One can see the undecidability of its state – is it true or false? But how would a self referential statement physically be formulated? Perhaps one would try a relay circuit, as shown:

![Figure 4](image)

Figure 4  In it’s on state, it would pull the middle plate down, thus turning it off. But then the plate would return to its default up position - turning it back on – and starting the effect all over again, ad infinitum. The hysteresis motion of the middle plate can be engineered to move very fast between the two positions (9). Physically, the relay’s state
would not truly be self referential - it still takes time for the middle plate to transition between the up/down positions, and a causal path can be obtained. This is explained through the Special Theory of Relativity, as the plate can never go faster than the speed of light. The example nevertheless intimates the idea of moving “in both” position states simultaneously. If we took this to the limit and imagined a system where the middle plate moved between both positions simultaneously, this would amount to a spacelike separation. See Figure 5. O would be one position and X would be other position. The time of moving between each plate “event” would be zero, but there would be a finite distance. Thus the spacetime interval \( S^2 = t^2 - x^2 \) would be \( 0 - x^2 \) which is less than zero. \( S \) would be \( ix \), where \( i \) is the \( SQR(-1) \).

![Figure 5](image)

Each plate state would be outside each other’s light cone of influence and could not affect the other. In effect, two separate circuits acting in two separate parallel universes so to speak would exist. But again, the physical limitations of the system prevent that spacelike situation from happening.

Quantum computing offers a physical situation whose explanation is compatible with such parallel universes. A physical example of a qubit (quantum bit) may be shown with a rubidium atom, which has a single electron in its outer orbit that may be excited into a higher orbit by shining a continuous wave polarized laser light for a set period of time (10). In basic digital computation, states of a bit are either 1 or 0. Because they are the only states, they may be treated as opposites of one another. In the rubidium example, the excited state is 1 and the lower state 0. A logical NOT gate applied to 0 gives 1, and this is physically produced by shining the pulse of laser light for the set time. But if you shine the laser for only half the time – a half pulse - the electron goes into a superposition of both the ground and excited states. What this means physically is if you attempted to measure the energy of the electron you would have a 50% chance of measuring the energy at the ground state, and a 50% chance of measuring it at the excited state. (Randomness thus factors in when trying to pinpoint the energy – and the specific state - one gets when one measures it, but the superposition itself is of both states).
Perhaps even more amazing, shining another half pulse puts it back to a definite state – the excited state. So two half pulses (call each h) together make a NOT gate. Thus $h^2$=NOT, and so each $h=SQR\ (NOT)$. Now, consider again the self referential situation. It is only when the state is considered with respect to itself that it is no longer itself. That’s the paradox. This Statement means the statement applied to itself, and Is False means NOT the statement. And thus $T^2$ effectively is a NOT gate, so that $T^2$= NOT, or $T=SQR(NOT)$, where $T$ is the statement. But we just saw the $SQR(NOT)$ corresponds to a physical operation – the half pulse. What this suggests is that a self referential operation may be represented mathematically as a superposition of digital states, and manifested physically through qubits, e.g. on a quantum computer. This makes intuitive sense because the state, after all, is undecided; furthermore the macro scale relay example adds support to interpreting the superposition as multiple universes. We also see the connection between the $SQR(NOT)$ and the analogous $SQR(-1)$. The power behind this is again the subtle understanding of digital states: that there are only two states forces one to represent the opposite of the other.

Remarkably then, the limit of computability is expanded when forced to consider the restrictions math and physics have on one another. Self referential operations, first seen to mathematically limit a physical computer, indeed can be the underpinning of qubit manipulation, and the physical foundation of quantum computing.

**Several concluding insights for further research:**

1) Applying yet another self referential operation should then correspond to another half pulse, which then takes the system out of superposition. This is equivalent to saying: [This statement (namely that this statement is false) is false]. But to consider that “this statement” is in fact the same full statement being referred to throughout, we could expand this as [(this statement that this statement is false) is false] is false. But $S$ = “this statement that this statement is false” is undecided. Therefore, to say $S$ is false is definitely not correct. ($S$ is false) is indeed a false statement. Thus, the self referential operation applied twice, $S^2$ indeed brings back a precise state of true, or 1.

2) The correspondence of a self referential operation with going to/from superposition suggests equivalence. Measurement of an operator causes the previous measurement of a complementary operator to be erased and thrown into superposition. Perhaps when a measurement is made, it sets up a new paradigm by which the complementary operator can no longer refer to itself and still be itself – it thus is thrown into a self referential scenario. But the operator being measured is taken out of superposition. Measurement thus swaps the way in which complimentary entities would refer to themselves – the
complementarity revealing limitations of those paradigms. The fact there are always complementary variables shows there will never be one measurement paradigm that can be used to explain everything else. Relating to Godel’s Incompleteness Theorem, this may show the idea that physical phenomena always requires explanation, that it can’t simply be reduced to a preconceived set of representations.

3) Max Tegmark takes the “physics is math” concept to the next level in the Mathematical Universe Hypothesis (MUH) and suggests that since there are an infinite amount of possible universes, anything mathematically possible is physically real and happened (11). Consider the self referential scenario: using MUH, we devise mathematically a universe that wouldn’t allow the physical explanation of an infinite multiverse. Such a universe could not exist via the explanation of MUH. But if it doesn’t exist, then MUH would assert this special universe once again exists. Taking this further, what concepts allow the universe(s) in which an infinite multiverse is valid? Perhaps it is mathematical Darwinism: only mathematical structures allowing an infinite multiverse explanation in fact then “live out” in an infinite number of physical universes?

4) Computation reduces to a sequence of steps; hence no matter the many forms of computation, the one factor that must be maintained is causality. Causality also relates to consciousness and experience. The simultaneity of different states in a superposition caused by a self referential scenario physically amounts to nonlocality. As Aharonov explains, randomness is needed to balance nonlocality if we are to maintain causality (12). Thus we see that the statistical nature of quantum mechanics is needed to balance the nonlocality brought on by a self referencing operation, to maintain causality. Perhaps this could be the basis of self awareness?
Appendix 1

Halting Problem – Easy Proof

The Halting Problem is:

Consider a program P and a string of symbols (or characters) S.

If P halts on S, then output 1 and if P goes into an infinite loop on S, output 0.

Theorem (Turing): There is no program to solve the Halting Problem.

Proof: Assume to reach a contradiction there exists a program Halt(P, S) that solves the halting problem - Halt(P, S) returns 1 if and only P halts on S.

Then, we could construct the following program A:

If Halt(x, x) then
   Loop Forever
Else Halt.
End.

Next, run the program A with input A on the Halt program. There are two possible situations:

a) Program A halts on input A. Since the Halt program by assumption will work, it returns 1 on input A, A. So A loops forever on input A, and we have a contradiction.

b) Program A loops forever on input A. Since the Halt program by assumption will work, it returns 0 on input A, A. So A halts on input A, and we have yet again a contradiction.
References


2) http://en.wikipedia.org/wiki/Ultraviolet_catastrophe


4) http://math.stackexchange.com/questions/91318/proving-the-cantor-pairing-function-bijective

5) Bourgrade, Keating, “Quantum chaos, random matrix theory, and the Riemann Zeta-function” Poincare Seminar


8) http://en.wikipedia.org/wiki/G%C3%B6del%27s_incompleteness_theorems

9) http://www.eecs.berkeley.edu/~tking/theses/iruchen.pdf

10) http://terpconnect.umd.edu/~sdebnath/Rubidium%2087.htm
