Does division of extension mean the same in mathematics as it does in physics?

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Abstract

In this essay, I attempt to raise and address some questions concerning the procedure of dividing extension mathematically, which is mental and the actual act of doing so, which is physical. By extension is implied a length, whether of matter or of a distance. And by dividing is implied the procedure of creating parts. Inevitably, such a discussion would touch on the continuity or otherwise of extended parts. I end the essay by moving the motion that we exorcise the lingering millennia old Parmenidean spell cast on our mathematics and physics and allow that whatever exists can perish.

Introduction

The question of the divisibility or non-divisibility of things to exhaustion has a checkered history with ideas traded back and forth over the ages between mathematicians, physicists and philosophers. However, till the present day, this question still has mathematical and physical aspects that appear unconquered. While historically at some point, the use of infinitesimals was assumed to resolve the problem mathematically, this was later superseded by indivisibles and again this was substituted with the concept of the limit. In modern times, the limit has again been questioned and the infinitesimals again resurrected in Non Standard Analysis (see a review [1]).

In accord with the theme of the FQXi 2015 Essay Contest, the discussion here will be organized by first raising a question, followed by answers as I feel would be given by the more mathematically-inclined, then as would be given by the more physics-inclined. The reader should be the judge of the better appealing answer, depending on his or her inclination. A two-part discussion, a hypothesis and a concluding remark follow and end the essay.

Questions and Answers

(i) What does it mean to divide extension?

Math: To divide is the mental act of creating parts of a whole length. In doing so, the created parts must sum up to the whole length. By this method, it is possible to find out how many times a given magnitude of length is contained in another.

Physics: To divide is the physical act of creating parts from a length. In so doing, the created parts are expected to sum up to the whole length. For quantities, like matter and energy, if evenly distributed along the length, the sum of the mass or energy of the parts must also equal the magnitude of the whole mass or energy. This can be expressed as a conservation law. There may however be phenomena in physics that can represent exceptions to this general rule. For instance, it is recognized that there is an equivalence between mass and energy, usually expressed by the famous equation, \( E = mc^2 \). If energy is regarded as a disturbance, it can be contemplated that for a material length of initially say 10 meters,
when divided, the parts may no longer sum up to 10 meters in length if part of the mass has been converted to energy.

(ii) What extension can be divided?

*Math*: All extension that has magnitude can be mentally divided. Zero (0) has no magnitude and so cannot be divided into parts. In the case of an infinite magnitude ($\infty$), while it may be divisible into parts, it will not obey the rule of the parts summing up to the whole. This being the case, it can be said that 0 and $\infty$ do not fall among extensions that can be divisible, but all in between can.

*Physics*: All extension that has magnitude can be physically divided, but only up to a limit. The reason will be given later in answering subsequent questions.

(iii) What can be used to divide?

*Math*: An extension can be divided by a line, which need not have any breadth. That is, a knife whose cutting edge is of zero breadth can divide into two parts a loaf of bread. In mathematics, discontinuity of the parts is not a requirement for division to be said to have taken place.

*Physics*: What can be used to physically divide must have some non-zero breadth so as to cause discontinuity between the parts lest the act of dividing a loaf of bread by a knife become physically futile. Also what can be used to divide must have a magnitude smaller than the extension to be divided in order that the act does not change from cutting to crushing. A source of disagreement then between math and physics is thus the question: can a zero-edged knife divide an extension? This leads to the next question.

(iv) Does the act of division require energy to be carried out? That is, is work done in the process?

*Math*: For work to be done, apart from the requirement that there must be some displacement, there must be action and reaction according to Newton's third law. Division of an extension by a line of zero breadth cannot elicit a reaction nor a displacement, so no work is done. Therefore, no energy is required for division to be carried out.

*Physics*: The act of dividing in physical reality will require energy, either in the physical cutting of bread, physical fission of atomic nuclei, biological fission of cells, etc. In all these, there is a discontinuity between the parts created by division. The parts become discrete entities and no longer continuous, with "something" separating them. That something may be the blade of a knife, cellular fluid, air or just space.

(v) Can division of extension be performed *ad infinitum*?

*Math*: The answer is a big 'Yes'. Firstly, a line of zero breadth can always divide a non-zero extension, no matter how small it is since the dividing line will always be smaller in magnitude. There is therefore no limit to how many times or how many slices the extension can be divided into as a result of the zero breadth of the mathematical knife. Secondly, no energy constraint exists. Thirdly, since the parts created remain in continuity there is no constraint of containing the parts created, as the spatial requirement is not higher than the initial extension to be divided. As such perpetual division of extension is mathematically possible.
Physics: The answer is a loud ‘No’. The act of physically dividing involves energy and the separation of parts created into discontinuous parts. There will therefore be a space constraint, apart from the energy limitation. For example, dividing a loaf of bread 50cm long into two parts of 25cm each, must have a separation between them, even if only a knife edge thick, e.g. 0.01cm, lest the act of division be a physical nullity. So we have an initial extended length of 50cm becoming 50.01cm (two halves 25cm each + 0.01cm). Even if the same knife is used repeatedly, air or space must remain between the discrete slices lest they revert back to loaf form making the division a non-event.

This represents another area of tension between math and physics. In math, division ad infinitum is possible, while in physical reality, division ad infinitum is constrained to the limits of the breadth of what is being used to do the dividing, the limits of energy availability and the constraints of space.

(vi) If division of extension into parts is finite, what can separate the resulting ultimately indivisible extended parts?

Math: Nothing can or needs separate the divided parts as has been discussed above. This applies even in the case where division is not continued perpetually but stopped after one or two cuts. Therefore, the act of division does not automatically imply discreteness. Since division cannot be exhausted, an extended length contains an infinite number of points and between any two points there always lies a third, and so on and so forth. There is no 'separation' between the points on a line and the points can be zero dimensional objects at the limit.

Physics: Something can and needs separate the parts at the limit of divisibility. For the case of extension in the form of matter, an accomplished act of division must be expressed in the matter being separated into parts made of matter, with the extension doing the separation between the divided parts being of a different nature. For illustration, when an iceberg in water is divided in two, liquid water does the separation between the parts. Although liquid water also has the property of extension, it is of a different nature to the extended iceberg. The extended iceberg cannot do its own separation into parts, and liquid water does the duty. We now know that water itself has a discrete nature in the form of the molecule, H₂O. Again, following the same reasoning the liquid water though extended cannot do its own separation into molecules, space does the duty of enabling the molecules have distance between them thereby enabling water exhibit its discrete nature.

Now, distance itself is an extension. Is distance capable of exhibiting a discrete nature with a limit to its physical divisibility? Considerable arguments for a minimum length scale exist in Quantum gravity theories. For a detailed review of what motivates the search for discreteness of length in physics, see [2], and the references therein. At about the Planck limit, ~ 10⁻³⁵m it is speculated that a minimum length exists. If indeed this is so, what does the separation of these non-zero fundamental minimum lengths into discreteness, since these fundamental units of distance cannot on principle do or be the source of their own separation into discreteness as has been argued above? We address this under discussion.

(vii) Where to cut?

According to Euclid's Elements, Book 1, definition 3, the extremities of a line are points [3]. Same therefore for segments, which are also lines. From the definition, lines can therefore be said to be
composite objects consisting of points, either finite or infinite in number. Although cutting of extended things is a common operation in physics and mathematics, (see for example Euclid's proposition 10, Book 1 of how to divide a line), an unresolved dilemma for both is the question of where on the line to cut.

Mathematically, there are an infinite number of points on the line, so there is always a point at the incidence of cutting and that point is indivisible and not amenable to be cut. So how then can the cutting operation proceed?

Physics does not fair better, even in the models where there is finite divisibility of length. In those models, there is a limit to the number of extended points or fundamental lengths, but there is also no distance between those points, since that distance will also consist of points. Further, one cannot resort to cutting at the boundary since as fundamental objects, both the boundary and its object are one and cannot be separate parts. So like, the case for math, where to cut is a problem.

**Can the dividing line and the divided line simultaneously occupy the same point?**

It is a common proposition in philosophy, that two places cannot occupy one place. This is justifiable on empirical evidence. It is also justifiable from Euclid's definition 1 in Book 1 [3], in that if there can be more than one point at a point, then at a point, there would be parts, thus contradicting the definition. Two points may not therefore coincide in the same place. To illustrate, a knife is a place, bread is a place, and knife and bread do not occupy the same place while interacting.

What then happens when the dividing line cuts the divided line since their points cannot coincide? Do the points on the divided line give way to allow passage of the dividing line? If that is the case, does that mean points can move from one place to another? If so, what does a point leave behind when it moves, is that itself not a place? Is there a place where the point is moving to? By this reasoning, it would be inescapable to avoid the problem of having more than one, if not a multitude of points at a point.

From the foregoing, I argue that our current ideas about what happens during the cutting of extension need more clarification, and I revisit them again for discussion under my proposed hypothesis.

**Discussion I. Distinguishing one part from others (with illustration)**

The difficulty of describing what can be said to be a separate part and what can be said not to be a separate part is the focus here, which we discuss with the aid of the diagram below.

From this diagram, we can infer that:

* The blue strips are discrete extensions since there is a yellow boundary between them. Had the yellow strip been of same nature as the blue, it cannot do the separation of the blue strip into discreteness, and the blue strip would have been continuous like the red strip. Thus for discreteness to be expressed, the separator must be of a different nature.

* Now let us zoom in on any of the boundary regions to see whether any fundamental physics can be hypothesized.
From what has been written earlier, physical division ultimately results in an indivisible and non-composite extension. As may be individually preferred, call these non-zero dimensional points, fundamental lengths or monads as Leibniz does. I prefer to call that fundamental unit, the point but this may breed misunderstanding since my point is extended in contrast to that of some mathematicians which is zero-dimensional. Such a fundamental unit of length is featureless, save that it is an extended thing. Lines, surfaces and bodies are composite things by Euclid’s definition, and color is a property of composite things. If extension is ultimately made of these non-composite, fundamental extended lengths which being indivisible, can in principle have 'no parts', such that we can talk of one end or extremity and another extremity, or an inner part and a boundary part of it, and if in principle there can be no available distance separating them, any distance itself being composed of the fundamental length, we therefore eventually confront the problem of how to distinguish one fundamental length from another on an extended line.

Let us describe things more graphically, put on a pair of special spectacles and peer at the multicolored strip in the diagram, screwing down the focus gradually towards the fundamental scale of length. What do you see? Initially, as you adjust the focus, everything you previously saw turns to a sea of different atoms, H, N, O, C, Fe, etc. No more blue, yellow or green strip. Then, as you turn the screw further, beyond the sub-atomic scale of quarks towards the fundamental scale of length, \( \sim 10^{-35} \text{m} \), the unexpected happens. At the scale of the fundamental building block, the blue extension is ultimately made of a featureless length, same with the yellow, green and red strips. At that fundamental color-blind scale, how is the boundary to be demarcated, say between one blue strip and another blue strip or between a blue and a yellow strip? At that scale, extension becomes continuous as there can be no space between the extended fundamental units of length. It seems what previously appeared to be Many things has been reduced in its fundamentals to One thing, suggesting that Many things could be an illusion. At the discrete level, the ultimate constituents of extended things become continuous and indistinguishable from one another!

If reality is fundamentally One thing, how then does nature play the trick of making a strip of One thing to present itself as a strip of Many things? These were concerns that agitated the minds of Parmenides of Elea, 515 BCE and his student, Zeno, who went further to propose paradoxes of motion in support of his teacher’s philosophy [4]. Till this day, the issues raised here have not been fully settled in physics and philosophy but more or less casually brushed aside since we are all living witnesses that many things exist and we see motion. But surely, Parmenides and Zeno were aware that there were many things and that there were moving things, but they were only trying to point at the shaky foundation on which these phenomena rested so that a stronger foundation on which they can rest is elucidated.

It appears that without the addition of some further ingredient into our physics, there are subtle nuances to what we call discrete and how it can be fundamentally expressed.
Discussion II

One major cause of the differences between mathematical and physical divisibility would appear to arise from the character of the 'dividing line' and not actually the divided line. In current mathematical interpretation, a dividing line can be of zero breadth. If it is desired to narrow the gap between divisibility according to physics and that according to mathematics, it may become necessary to depart from the current idealization of a line as having no breadth to a line having a breadth of at least the minimum extension.

Lines have a manifest physical reality of at least one dimension, even in Platonic interpretations. If a dividing line is now mathematically endowed with some minimum breadth and possibly some thickness too, probably of Planck dimension, then while it can divide extension, it cannot do so beyond the size of its breadth and some harmony can be restored between math and physics on the questions surrounding division of extension.

The 'divided line' presents us also with its own trouble of where on it can be cut without leading us into contradiction as has been earlier pointed out.

Lastly, there remains the question of what can separate one fundamental length from another, lest they retain their continuity as one continuous length. Distance cannot separate them, being also made of points. For a discrete nature for space to be expressible, points cannot be the separator of points. We are thus confronted with a duality. Extension can be composite and discrete, being not infinitely divisible on the one hand, and on the other hand it can be continuous, one discrete, minimum length not being distinguishable from another.

Being of a different nature, I next propose a hypothesis of time as the separator of minimum lengths, enabling the physical manifestation of discreteness in otherwise 'syrupy' space. This hypothesis also suggests how using the same mechanism, a line can be divided without presenting us with the dilemma of where to cut.

A hypothesis

"Give your mind now to the true reasoning I have to unfold. A new fact is battling strenuously for access to your ears. A new aspect of the Universe is striving to reveal itself. But no fact is so simple that it is not harder to believe than to doubt at the first presentation" – Lucretius, 55 BCE in De Rerum Natura

Borrowing from cosmology and quantum physics, space appears not to be an eternally existing entity. From some cosmological models, space was previously non-existent, then appeared at a minimum length (Big bang), is currently expanding and possibly at a future time, it would collapse, reduce to the minimum length and disappear (Big crunch). The picture depicted in the cosmological scenario is that of a line which was previously non-existent, becoming a point and extending till some future time when it would reverse, shorten to a point and disappear entirely. Of course, not everybody accepts the cosmological theory, so we are assuming its correctness.

In Quantum mechanics, an object, e.g. an electron at a location in the atomic orbit can without seeming to traverse the 'space in between', arrive at a different location, the so called 'quantum jumping'. In a way this could be imagined as suggesting the possibility of the disappearance of that
'space in between', separating the electron orbitals. For analogy, a place called Lagos is some distance away from a place called New York, and a traveller in Lagos can ordinarily not get to New York without passing the space in between. But suppose that intervening space between Lagos and New York disappears, as magical and unfamiliar as it sounds? The result is the traveller actually arriving New York without even leaving the place in Lagos, where he was.

The take away proposition from this is that extended points may not be eternally existing physical objects. That being the case, the different durations of existence can be the fundamental mechanism conferring discreteness to a seemingly continuous line. In other words, 'time' is the separator and conferor of discreteness on 'space' and removes the earlier pointed out illogicality of space doing its own separation into discreteness.

From this I make the following postulate:

**the non-zero dimensional point does not have an eternal existence, but can appear and disappear spontaneously, or when induced to do so.**

I now bring this postulate to bear on the difficulties surrounding divisibility of extension and what it means to cut; the problem of distinguishing one from another and how come Many from One; and some of the paradoxes of motion.

I crave the reader's indulgence here to temporarily set aside current doctrine on the eternal existence of points, either of the mathematical or physical variety and not ask like Parmenides, "How could what is perish?"[5]. For the moment, if only for amusement, indulge in a 'what if what is can perish?' What will our physics look like if points can appear and disappear?

The act of cutting: In mathematics, we have shown that no actual discontinuity takes place since the dividing line has no breadth, and in physics, even if the dividing line has breadth it must encounter an extended point, which since it is indivisible and cannot be divided into parts, will not allow cutting to take place, neither will it give way since what it will leave behind is also a point. In this hypothesis, what actually happens when things are physically cut and divided?

If the dividing line causes at the point of incidence the creation and then annihilation of an extended point on the line to be divided, such that on annihilation the dividing line becomes temporarily interposed between the segments, the extension becomes divided. There is no occupation of one place by two points, and the intractable difficulty of 'where to cut' in mathematics and physics is avoided.

Creation and annihilation of space follow the application of force in this hypothesis. That is, cutting of matter results in the creation of space between the divided parts and the simultaneous disappearance/destruction of space without (see the knife and loaf of bread below). Points in the direction of the cut are annihilated and cutting becomes a physically possible act.

"How could what is perish? How could it have come to be? For if it came into being, it is not; nor is it if ever it is going to be. Thus coming into being is extinguished, and destruction unknown": (B 8.20–22)
**Many and One:** The whole of existing space is one fundamental length continuous with the next. Ultimately, if extension cannot be its own separator into discreteness, the hypothesis proposed introduces 'time' as the separator of extension into discrete. By 'time', I mean duration of existence, i.e. extension can start to exist and cease to exist and as all minimum lengths do not have the same life span, the discrete nature of otherwise syrupy space becomes manifest. A rhythmic and peculiar appearance and disappearance by the fundamental lengths in a particular composite location will confer qualities associated with composite things on that particular location. Such ON and OFF behavior can be characterized by the binary digits, 0 and 1, as first alluded to in my essay entry in the FQXi 2013 contest [6]. Hence, despite an extended multi-colored strip, being fundamentally composed of the same thing, the peculiar rhythm of the fundamental lengths confers on each segment the quality associated with blue, yellow, green, etc, so that Many can arise from One, and One from Many.

Color is used here for example but other physical characteristics of composite things, like mass and electric charge can be said to be derivatives of some behavior at the fundamental length scale. In a sense, we could ponder if this annihilation and appearance of places called points could be our sought after fundamental event.

**Phenomenon of motion:** The familiar phenomenon of motion and the associated paradoxes may serve to strengthen this hypothesis. Already, there are suggestions from General relativity that a straight line can be curved if brought into proximity to a gravitational mass. Only a composite entity can so change its configuration, and therefore we can remain comfortable in the belief that a line is a composite thing with a physical reality. Again, although mainly applied to hyper-fast travel, it has been suggested that General relativity permits the compression of space-time in the direction of motion, in the so-called Alcubierre drive [7]. In other words, in General relativity, the line between a moving spacecraft and its destination may not retain its configuration in the direction of motion. The hypothesis here extends this idea to all motion, not just hyper-fast travel.

In one of Zeno's paradoxes, the Dichotomy Argument, Atalanta, the runner in order to reach his destination, must first travel half the distance, and on getting there, travel half the remaining, and on getting there travel half of that remaining, etc, with the result of almost reaching his goal but never finitely getting there. Mathematical solutions have come to Atalanta's rescue, as discussed in [4]. Among these are (i) The limit. In this proposal, the distance to destination tends to zero, (ii) The infinitesimal. The distance between Atlanta and his destination eventually becomes an infinitesimal quantity, smaller than any real measurable number. Although not exactly zero, it is indistinguishable from the zero destination and (iii) The Cauchy solution. The sum of half distances of a whole distance equals to the total distance, i.e. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1$ (the total distance).

In the Cauchy solution, Atalanta somehow reaches his goal and not just tend towards it as in the first two solutions. However, this solution does not reveal what that last fraction in the series will be. Should there be a minimum possible length in physics, this solution also leaves us helpless in confronting the paradox.
If we however entertain the possibility that as Atalanta 'runs', points along that line perish sequentially, in a $dx/dt$ fashion, Atalanta finitely reaches his goal in a finite time and the last $dx$ to disappear is the last extended point at the destination. It may be noted here as an aside that Atalanta does not actually leave his place in reaching this destination. It is the space between Atalanta and his destination that disappears, while space appears between Atalanta and the origin. Perhaps, this may have led Zeno to further say in his other Arrow paradox, "What is in motion moves neither in the place it is nor in one in which it is not" [8]. In other words, motion definitely occurs but not by a composite thing leaving its points behind in exchange for another set of points.

The reader may at least consider for amusement that when you take a walk from one end of the room to the other, you cause the perishing of the extended points between you and the wall towards which you walk, while simultaneously causing the creation of extended points between you and the wall away from which you are walking, while the fundamental lengths of which you are a composite never actually leave their place.

Current mathematical doctrine does not appear to have allowance for what is to perish. For example, a set member in set theory cannot perish. In physics, on the classical scale, the perishing of an existing thing out of a large number may not have a statistical significance, but in quantum physics this could mean a lot to phenomena on that scale. Despite this hypothesis being against the doctrine of eternally existing points, does this trick resolve existing paradoxes of motion and geometry? Is this hypothesis falsifiable by real or thought experiments? Is it impossible? If what is can perish in cosmology and what is can subsequently perish in quantum fluctuations, what then could be sacrosanct in a point not perishing? Perhaps, if Parmenides had allowed for the possibility that what exists can perish, he would have told us much earlier how Many could come from One and his student, Zeno would not have had any reason to formulate his paradoxes, since distance can perish. I certainly look forward to a falsification of this possible trick of nature by dialectic or experiment.

"To my mind there must be, at the bottom of it all, not an equation, but an utterly simple idea. And to me that idea, when we discover it, will be so compelling, so inevitable, that we will say to one another, 'Oh, how beautiful. How could it have been otherwise?'" – J.A. Wheeler, interview with Timothy Ferris, Coming of Age in the Milky Way, p. 346

Concluding remarks

Even though the procedure of division is known to both mathematics and physics, we argue that it does not always connote the same meaning. In physics, there appears to be a limit to what it means to be divisible in a practical sense, while in mathematics divisibility ad infinitum is a viable procedure. These discrepancies in meaning may underlie some of the paradoxes involving divisibility in physics, if it is chosen to interpret a physics procedure, with a purely mathematical language. On the other hand, given their twin development, perhaps mathematical language and axioms can be improved so that it can more completely describe physics. Finally, I suggest here that without some further ingredient in our understanding, the act of division of extension will be impossible whether mathematically or physically. One solution to ponder is the idea that the extended point may have a temporal existence. I therefore move the motion that we exorcise the lingering millennia old Parmenidean spell on our mathematics and physics and allow that what exists can perish. Nothing is ultimately conserved. Any seconders?
References


