Imagination and the Unreasonable Effectiveness of Mathematics: 
A Case Study in Quantum Mechanics

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March 4th, 2015

Abstract

At least since a famous 1960 paper by Wigner, the effectiveness of mathematics in the natural sciences has been the subject of ongoing debate. This paper argues that if we take nature to be consistent, then given that we have complete freedom short of inconsistency in choosing axioms to mathematically model reality, we should expect mathematics to potentially be “unreasonably” effective in modeling reality. The reason why it actually turns out to be so, however, is attributable to human imagination.

As a case study to illustrate this, I present highlights of recent work which attempts to connect the foundations of mathematics to the foundations of quantum mechanics by means of a tentative novel axiom to be added to ZFC set theory. This axiom is meant to formally introduce into mathematics the intuitive distinction between actualities and potentialities by permitting the construction of novel mathematical objects which are in a certain sense incomplete. These objects exhibit quantum-like features, and a theorem directly connects them to the Feynman path integral for the simplest possible case.

Keywords: Actualizability, actualization factor, context, default specification, effectiveness of mathematics, emergence specification, empty element, e-spec, incomplete embedding, incomplete ordered pair, incomplete relation, incomplete space-time vector, pro-actuality, pseudo-nonlocality

1 Introduction

Anyone who studies mathematics and physics at least in some depth will inevitably come across connections between them which seem surprising. Indeed, in 1960 Eugene Wigner wrote a famous paper, The Unreasonable Effectiveness of Mathematics in the Natural Sciences [1], in which he pointed out that mathematics seemed to be far more successful in helping model aspects of reality than might have been naively expected. Two aspects of mathematics highly relevant to this issue are:

1. The freedom to choose one’s starting assumptions in building a mathematical representation of a system.

2. The requirement that the system of propositions derived from these assumptions be consistent.

It is easy to fail to appreciate just how permissive the freedom to choose one’s axioms in mathematics really is. To give a rather silly example, there is nothing that prohibits the construction of a mathematics in which axioms, instead of referring to sets or numbers as primitives, refer to, say, beetles. The reason we don’t do this is not a lack of freedom, it is a lack of usefulness: beetle-based mathematics seems ill-suited to express mathematical relations between things which are not beetles.

The freedom, however permissive, stops short of that of choosing axioms which are inconsistent with each
other. To be sure, the standard of consistency here is not the standard of proof but of avoiding ‘obvious’ inconsistencies. In fact, by a well-known theorem of Gödel, any sufficiently complex mathematical system cannot prove its own consistency. Zermelo-Fraenkel Set theory with the axiom of choice (ZFC) is generally regarded today as the foundation of mathematics, yet it is not known with absolute certainty that it is consistent. The requirement of consistency ensures that if mathematics models a system that is itself consistent, then mathematical axioms which reflect some fundamental aspects of that system will necessarily lead to propositions that are also reflective of aspects of that system. Thus we are led to the question: Are the laws of nature consistent? We have no hard evidence that they are not. Indeed, it would seem that inconsistencies in the laws of nature might have been by now exploited by some clever enterprising individuals to create perpetual motion devices, time machines which permit you to go visit your grandfather before your parents were born, or other whim-whams. We see no such things, and so far it has been an extremely safe bet that nature is in fact consistent.

So if we take the laws of nature to be consistent, and grant complete freedom short of inconsistency in choosing our axioms, then of course mathematics should be “unreasonably” effective in modeling reality: It should be so because it should be unreasonably effective in modeling any consistent system. The freedom to choose any consistent set of axioms entails the freedom to choose exactly those sets which lead to the most effective mathematical representations of a given consistent system, and the requirement of consistency guarantees that how propositions are derived in those models mirrors relationships in the system itself.

However, this does not explain why mathematics actually is as effective as it has been in modeling reality, for even if there should be “unreasonably” effective mathematical models of reality, it is by no means given that we would discover them. In order to find the answer to this question, we must look beyond mathematics. The ‘secret sauce’, the power that drives the effectiveness of mathematics is...human imagination. Discourses on the triumphs of imagination are far more likely to mention the music of Beethoven or the literature of Shakespeare than the geometry of Riemann or the calculus of Newton, but imagination is just as important in mathematics as it is in any other field and, arguably, under some circumstances, even more so. Abstractly, this claim is unlikely to convey just how strong the influence of imagination on the effectiveness of mathematics really is. For this reason, I would like to now take you, dear reader, on a journey during which I endeavor to show in action how much a little imagination can power the effectiveness of mathematics in modeling reality. I will do this by using my recent research, which attempts to connect the foundations of mathematics to those of quantum mechanics, as a case study, and which I fancy uses a modicum of imagination.

2 The Default Specification Axiom

Everyone understands at an intuitive level that existence as a potentiality is distinct from existence as an actuality. When I have a die in my hand, the outcome of a throw not yet thrown does not enjoy the same ontic status as the outcome of a thrown toss. Yet the state of early 21st century mathematics is such that everything represented by mathematics is represented as an actuality. Representing inequivalent aspects of reality as if they were equivalent is bound to create problems, but sometimes these can be compensated for by the freedom to choose our assumptions, in this case our assumptions about what sort of a reality the mathematics models. If we want to take the implication of the abstract mathematical representation that the two outcomes are equivalent literally, we might posit a reality such that after the die is thrown, our universe “splits” into six different versions, one associated with each possible outcome. Since there is nothing special about this particular throw, we would then have to treat all outcomes of every event with more than one possible outcome the same way, and this would lead directly to a mathematical model of reality most accurately described as a tree-like multiverse with rapidly proliferating branches.

Instead of going down that road on our journey, let us use some imagination to introduce the distinction between actuality and potentiality, already intuitive to us, formally into mathematics in order to model fundamental aspects of reality in a novel way. The distinction is introduced by means of what I call the default specification principle:

The absence of an explicit specification entails all possible default specifications.

Applied to the above example it says that as long as I do not throw the die and therefore do not have an actual outcome of a throw (the absence of an explicit specification), then all possible outcomes exist merely as
potentialities (all possible default specifications).

There are two hurdles to formally incorporating this principle into mathematics. First, classical logic, which lies at the heart of mathematics, is not equipped to transcribe phrases like “the absence of” which fail to refer (the logical transcription is ¬∃x(ι = t), but free logic is [2][3]. In what is known as positive free logic, terms which fail to refer are associated with an outer domain, which is segregated from the inner domain, reserved for things that satisfy the criterion of “existence”. In the outer domain, the counterparts to the quantifiers ∃ and ∃ are Π and Σ, respectively. Using this logic one can define two important new mathematical tools:

- **The empty element**, (“naught”), symbolized by ∅ (usually ∅ denotes the empty set but now it is given a new meaning), and defined as the absence of an element such that

  \[
  \{∅\} = \{
  \]

  where \{\} is the empty set.

- **A context** which can be thought of as the image in the outer domain of a given set Z in the inner domain.

  It is symbolized by adding the subscript C to the given set. Since it contains no elements of the inner domain it is really the empty set, but with the additional intension that it is considered a subset of Z.

The second hurdle is that the principle is modal, distinguishing between “explicit” and “default” specifications, translatable in this context as “actual” and “actualizable”, respectively, where the latter term is strictly used in the sense of a potentiality due to the absence of an actuality. This hurdle can be overcome by extending classical logic to modal logic [4]. This logic features modal operators, typically symbolized by □ and ◇, which qualify logical sentences in a way that, for any sentence P, obey the following duality:

\[
□P \iff \neg \neg P
\]

The best-known modals are “necessarily” and “possibly”, and these are used to build two new operators, which however do not obey that duality:

<table>
<thead>
<tr>
<th>Operator</th>
<th>Alethic Modal Interpretation</th>
<th>New Operator</th>
<th>Actualizability Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>□P</td>
<td>Necessarily P</td>
<td>□P = (□P &amp; ¬¬P)</td>
<td>Pro-actually P</td>
</tr>
<tr>
<td>P</td>
<td>Actually P</td>
<td>P</td>
<td>Actually P</td>
</tr>
<tr>
<td>◇P</td>
<td>Possibly P</td>
<td>◇P = (◇P &amp; ¬¬P)</td>
<td>Actually P</td>
</tr>
</tbody>
</table>

**Pro-actuality** is a condition in which only one actualizability can become actual. The formal system to which these are meant to be applied is known as VK. Taking the required logical structures to be in place, the default specification principle can be formulated as a set-theoretic axiom:

**Axiom of Default Specification (Axiom D)[Tentative]**

\[
\forall Z \forall a(a \in Z \Rightarrow \Sigma Y \Sigma x(Y = Zc \land x = a \land x \in Y \land (a \in Y)))
\]  

The axiom is labeled as “tentative” because, this being a work in progress, I have not yet carried out all the necessary work to ensure that it leads to all the desired results and does not introduce any inconsistencies; however, for the purposes of this paper, I will treat it as non-tentative. In words, it says that for any set Z its empty subset serves as the context in which the elements of Z are actualizable (i.e. in the outer domain) its members. Added to ZFC set theory it yields a free modal extension I call ZFCD.

If consistent, then ZFCD is powerful enough to formally represent the intuitive distinction between actuality and actualizability. For example, the set of outcomes of the unthrown die mentioned above is actually (short for: it is actually the case that it is) the empty set, but actualizably the set containing six possible outcomes as members, unless the die is loaded to always land on the same face, in which case it is pro-actually the set containing only that outcome as a member.
3 The Actualization Factor

The new axiom is meant to permit the construction of novel kinds of mathematical objects inexpressible in ZFC which have in common the feature that they are incomplete. We begin with an incomplete ordered pair \((x, \emptyset) \in A \times B\) or \((\emptyset, y) \in A \times B\) for some sets \(A\) and \(B\), which by axiom D manifest themselves (for instance, when represented graphically) as the superposition as actualizabilities of all ordered pairs which are elements of \(A \times B\) that have the actual component in common. We can extend this to any incomplete \(n\)-tuple recursively, and sets of these will be called incomplete relations. For example, the set of all third-component incomplete triples \(\{(x, y, \emptyset)\}\) is the incomplete relation any element \((x, y, \emptyset)\) of which by axiom D manifests itself in \(R^3\) as an infinite line parallel to the z-axis (fig. 1a).

![Figure 1: Completion of \((x, y, \emptyset)\) to \((x, y, z)\)](image)

This is an instance of an incomplete embedding of \(R^2\) in \(R^3\): \(x\) and \(y\) coordinates in \(R^2\) are mapped to \(x\) and \(y\) coordinates in \(R^3\), but the \(R^2\) plane is not mapped to any \(z\)-coordinate. Thus, \((x, y, \emptyset)\) can be thought of as a vector that “lives” in \(R^2\) but is “observed” in \(R^3\) (compare to the concept of Dimensional Frame of Reference in [5]). As shown in fig. 1b, \((x, y, \emptyset)\) is completed to \((x, y, z)\) by means of what I call an emergence specification (e-spec), an element \(\{(x, y, \emptyset), (x, y, z)\}\) of the map from the incomplete relation \(\{(x, y, \emptyset)\}\) to \(R^3\) which “collapses” the superposition of actualizabilities to permit the emergence of the actual element \((x, y, z)\).

An uncompleted actualizability with an e-spec is a pro-actuality, which upon completion turns into an actuality. But what if there is a collapse of actualizabilities to an actuality in the absence of an e-spec? Then we apply axiom D, obtaining a superposition of all possible e-specs as actualizabilities, and associate a measure with the subset of the domain of the emergence map that is the context, and which tells us how much to expect each e-spec to complete the actualizability relative to all the other e-specs. Thus, the context can be interpreted as a sample space and the measure associated with the “collapse” of actualizabilities (i.e emergence of an actual outcome) in the absence of an e-spec is just what we call probability. This measure is different from non-probabilistic measures because it is over a set in the outer domain, and thereby captures the concept of probability. To illustrate this for the continuous case, consider again the incomplete vector \((x, y, \emptyset)\). The integral over the infinite line of actualizable points parallel to \(z\) is

\[
\int_{-\infty}^{\infty} Adz
\]

where \(A\) will be called the actualization factor, a quantity which indicates that any integral under which it appears is an integral over actualizabilities (i.e. elements in the outer domain), not actualities. But now suppose we impose the following constraints:

\[
1 = \int_{-\infty}^{\infty} Adz \quad 0 \leq \int_{a \geq -\infty}^{b \leq \infty} Adz \leq 1
\]

It is clear that with these additional constraints the integral becomes a probability and the actualization factor reduces to the more specific concept of a probability density. We will now turn our attention to quantum mechanics.
4 Contextuality as a Manifestation of Incompleteness

Contextuality in quantum theory refers to the dependence of the outcome of a measurement on the particular arrangement in which it is carried out along with other measurements. It was put on a formal mathematical footing by Kochen and Specker [6], and the idea behind it was later expressed more simply by means of what is now called the Peres-Mermin Magic Square [7][8]. To appreciate its apparent counterintuitiveness, consider the analogy of a 3 × 3 array of the numbers 0 and 1, leaving the value of the ninth field unknown. The objective is to determine whether the ninth field should have the possible value of 0 or 1 given the following rules: 1) the row-wise sum should be odd and 2) the column-wise sum of the numbers should be even. A quick check reveals that it is impossible to satisfy both rules for any arrangement of 0’s and 1’s with a single number. In the given array, by rule 1), the ninth field should have a value of 1, and by rule 2) it should have a value of 0. For numbers, the two rules are mutually exclusive.

Yet in quantum mechanics, something analogous to two identical copies of the same arrangement each complying with a different rule is possible: performing a set of measurements analogous to performing the column-wise sum yields the analog of the value 0 for the ninth square, and performing measurements analogous to performing a row-wise sum yields the analog of the value 1 for that square. Quantum mechanical properties are contextual.

With our new mathematical construction it is easy to replicate this phenomenon directly in the analogy: Let the value of the ninth field be represented by $\emptyset \in \{0, 1\}$. Notice that this means that the 3 × 3 array is in fact incomplete.

By axiom D, the ninth field “slot” is associated with the superposition of the values 0 and 1 as actualizabilities (fig. 4).

Figure 2: A simple analogy of the Mermin-Peres magic square

Figure 3: Replacing the value of the ninth field by $\emptyset$ yields an incomplete 3 × 3 array

Figure 4: By axiom D, $\emptyset \in \{0, 1\}$ implies a superposition of two actualizabilities at the empty slot
Now we need a set of e-specs that specify the following: Let the emergent value be that which has the same parity as the column-wise sum and the opposite parity as the row-wise sum of the other two terms. Since in this example the values of both the two fields above and the two squares to the left of the ninth field sum to an even number, a column-wise addition collapses the superposition of possibilities to 0, and a row-wise addition from left to right collapses it to 1.

![Figure 5: The superposition “collapses” to a value that depends on the order of operation](image)

I used e-specs in the analogy, but the quantum mechanical situation involving commuting spin operators also uses a set of e-specs, though this tends to be obliquely buried in the mathematics of tensor products. It is called “conservation of angular momentum”; conservation laws are the emergence specifications of nature. Our set of e-specs can also be cast as a “conservation law”, namely conservation of column-wise even parity and row-wise odd parity of the sum of three terms. Thus, contextuality is the hallmark of incompleteness.

## 5 Pseudo-nonlocality in Euclidean Space

Another seemingly strange feature of quantum mechanics is called nonlocality, though pseudo-nonlocality may be a better term for reasons we will see shortly. The demonstration of this effect typically involves two or more quantum objects which are said to be entangled. This means that they are described by the same quantum state for the whole system in such a way that the state of one individual member of this system cannot be described independently of the states of the other members. Recall that the quantum state is a superposition state, so in this situation it describes a superposition of various possible arrangements of the combined states of the members. The simplest case is that of spin: Here the quantum state of a two spin-\(\frac{1}{2}\) particle system is a superposition of the states (particle 1 has spin up, particle 2 has spin down) and (particle 1 has spin down, particle 2 has spin up). A measurement of the spin of one of these “collapses” the superposition of the system: Say, if the spin of particle 1 is measured to be down, then we know immediately that if we perform a spin measurement on particle 2, we will find it to be up. The quantum correlations between the particles go beyond what is possible in any classical arrangement, and this was proven in a theorem by Bell.

There is an important subtlety here to consider, namely, it is not the case that a measurement on particle 1 also constitutes a measurement on particle 2 (If it did, then that would amount to true non-locality). This point is often misunderstood, but we know that the state cannot have been measured by a measurement on the other entangled particle because if that were true, then for properties that evolve according to the Schrödinger equation, the measurement of the state of particle 2 by a measurement on particle 1 would mark the beginning of the time evolution of the state of particle 2 from its measured state, which inevitably leads to a superposition state. This would imply that there exist time periods during which a measurement on particle 2 after a measurement on particle 1 should at least sometimes lead to the observation of violations of the quantum correlations, since only one state (the initial one) in the superposition gives the right correlation. This is incompatible with experimental evidence, which to my knowledge has always confirmed quantum correlations. Hence we are only permitted to state that if a measurement is performed on particle 1, then we know that only if a corresponding measurement is performed on the entangled particle 2, then the outcome will be found to be correlated.
Now, to construct our analogy in $\mathbb{R}^3$, consider a system $S$ of two incomplete vectors, $v_1 = (x_1, y_1, 0)$ and $v_2 = (x_2, y_2, 0)$, and suppose that due to the absence of a single e-spec out of two possible ones, Alice is equipped with the set of two e-specs $\{v_1, v_{1A}\}$ as potentialities, where $v_{1A}$ and $v_{2A}$ are the completions of $v_1$ and $v_2$, respectively, with the coordinate, say, $-z_0$. They are underlined to indicate they are actually vectors in $\mathbb{R}^3$. Also suppose Bob’s set is $\{v_2, v_{2B}\}$, where $v_{1B}$ and $v_{2B}$ are the completions with $+z_0$ (see fig. 6a).

![Figure 6: Alice’s “measurement” of $v_{1A} = (x_1, y_1, -z_0)$ permits her to predict with certainty that only if Bob performs a measurement, will he obtain $v_{2B} = (x_2, y_2, +z_0)$ even though she did not measure it herself.](image)

Since there are two e-specs for each, if either of them attempts to complete an incomplete vector in what we will call a “measurement”, it is equally likely that either $v_1$ or $v_2$ will be measured. Thus, Alice, for instance, cannot predict whether the outcome of her measurement will be $v_{1A}$ or $v_{2A}$, but once she performs it, she will immediately know that only if Bob performs a measurement on $S$, he will actually obtain the vector with the other subscript number. (fig. 6b). Notice how this correlation features the same subtlety as the quantum correlations: Alice’s measurement of, say, $v_{1A}$ which led to the “collapse” of the infinite line manifestation of $v_1$ did not do likewise for $v_2$, but nevertheless she knows with certainty that if Bob measures $S$, he will obtain $v_{2B}$, no matter how far away he is. In short, after Alice’s measurement, the actualizability of $S$ turns for Bob into a pro-actuality, but for it to become an actuality Bob still needs to measure it.

Locality can be defined in terms of the constraint that no matter, energy, signal, information or causal influence can travel faster than the speed of light [9]. The setup gives the appearance of an immediate transmission of something because the actual vectors emerged “far away” in $\mathbb{R}^3$, yet were correlated. The correlation is due to the fact that the incomplete vectors out which they emerged are part of a single system which is not an object in $\mathbb{R}^3$! By this I mean that the metric relations and the transformation properties that intrinsically characterize it are those of $\mathbb{R}^2$, not $\mathbb{R}^3$, since the embedding is incomplete. Failing to realize this gives the appearance of non-locality, but since in fact this correlated emergence phenomenon involves the transmission of nothing, a more appropriate term for it is pseudo-nonlocality.

## 6 The Physical Origin of the Feynman Path Integral

So far I have demonstrated quantum-like effects only with analogies, but unless these effects are directly connected to the quantum theory proper, they will have only limited persuasiveness, and rightly so. The burden of showing that their similarity to quantum phenomena is far more than mere coincidence rests with the one making that assertion. I now intend to meet that burden.

Consider the incomplete spacetime vector $(r, \theta) = (r(t_f) - r(t_i), \theta) \equiv (r_f - r_i, \theta)$, where $\theta$ replaces coordinate time $t$. How would it manifest itself to us? To simplify this problem, let me consider the $1 + 1$ dimensional analog $(x_f - x_i, \theta)$ in the non-relativistic limit, which represents two points $x(t_f)$ and $x(t_i)$ in space and time not connected by a worldline but still “connected” since they are part of the same incomplete vector. It is important to note that $t_f$ and $t_i$ are not associated with the object described here, but are just coordinates that locate the points $x_i$ and $x_f$ in time.
The key here is to realize that since non-relativistically time foliates “all of space at once” in each instant, and the incomplete object lacks any characterization in terms of time, by axiom D the actualizable manifestation will “fill” all of space at each instant between \( t_i \) and \( t_f \). To implement this realization mathematically, we divide the time interval between \( t_i \) and \( t_f \) into \( N \) small increments \( \Delta t \) and define \( x_p \Delta t = x_n \). We wish to integrate over the ways that take us forward in time from \( x_n \) to every space coordinate at the next instant, \( x_n \). Since this is an integral over actualizabilities, we need to associate each contribution during the \( n \)th time segment with the actualization factor, symbolized by \( A_n \). Two possible steps from \( x_0 \) to \( x_2 \) for a sample path are shown below:

![Diagram](a) A single path from \( x_0 \) to \( x_1 \)  
(b) A path from \( x_0 \) to \( x_2 \) via \( x_1 \)

\[
\text{Figure 7: Going from } x_0 \text{ to } x_2 \text{ via a single path}
\]

Integrating over going from \( x_0 \) to all possible \( x_1 \) points yields \( \int_{-\infty}^{\infty} A_1 dx_1 \), and integrating over going from \( x_0 \) to all possible \( x_2 \) points yields \( \int_{-\infty}^{\infty} A_1 dx_1 \int_{-\infty}^{\infty} A_2 dx_2 \). Integrating over going from \( x_0 \) to \( x_f \equiv x_N \) and taking the limit \( N \rightarrow \infty, \Delta t \rightarrow 0 \) yields

\[
\lim_{N \rightarrow \infty, \Delta t \rightarrow 0} A_N \prod_{n=1}^{N-1} \int_{-\infty}^{\infty} A_n dx_n
\]

Where \( A_N \) is not under an integral because the last step involves getting to only a single point in space. Up to now, no mention was made of quantum mechanics, but the connection is supplied by the following

**Theorem.** If \( A_n \) is identified with the free-particle transition probability amplitude \( \langle x_n | x_{n-1} \rangle \), then equation (6) yields to first order the free-particle Feynman path integral in \( 1 + 1 \) dimensions.

**Proof.** see technical endnote. \( \square \)

Although proven only for the free-particle case, given the distinction introduced here, the theorem implies that the Feynman path integral is an actualizable object (because the addition of a potential presumably does not affect the path integral’s ontology), and since it can be directly related to the quantum state by considering it as the propagator \( K(x, \dot{x}, t) \) which takes the initial state \( \Psi(t_i) \) to some future state \( \Psi(t_f) \), the quantum state must also be actualizable:

\[
\Psi(x, t_f) = \int K(x, \dot{x}, t)\Psi(x, t_i)dx'
\]

where it is the phase factor which permits us to write \( \Psi(x, t) \) as a function of time even though the underlying object has no (spacetime) time coordinate. The completion

\[
(x, \emptyset) \longrightarrow (x, t_M)
\]

gives as the fundamental ontology behind a position “measurement” the emergence of the state of an actual spacetime object out of the superposition of states of actualizable objects at the instant \( t_M \) it is measured:

\[
\Psi(x, t) \longrightarrow \psi(x, t_M)
\]

Where the eigenstate \( \psi(x, t_M) \) is underlined to mark it as the state of an actual spacetime object at instant \( t_M \) (also see ).
Let me now address why in section 3 the actualization factor was reducible to a probability density but in the theorem had to be considered to be a probability amplitude. Here is the explanation: In the absence of an e-spec (i.e. in the absence of a conservation law which permits a spacetime object to emerge in only one possible way), we might indeed expect the “collapse” of the actualizabilities to an actuality to be probabilistic. However, there is a slight complication: As the quantum state represents an object characterized by the intrinsic absence of t in spacetime, it is also characterized by the absence of the directionality of t. By axiom D this implies a superposition as actualizabilities of the forward and backward time directionalities. The time directionality is given by the sign of the imaginary exponent of the quantum phase factor. Since each directionality contributes in the exponent only half the full representation of all actualizable states in a particular region of space, the full representation is given by the product of the quantum state and its complex conjugate i.e. its absolute square, and it is this which is subject to a probabilistic collapse over a given region of space. What I just described amounts to nothing other than the Born rule.

The free particle case, being the simplest, allows one to go furthest without a more fundamental physical theory at hand. I previously proposed such a theory, and it directly motivated the development of the mathematical ideas highlighted here [10] [11]. I call it the dimensional theory, and probably the most accessible introduction is via a talk I gave at a quantum foundations conference in 2012, at which time I did not even dream of such a mathematical foundation [12]. I should caution you that as it is extremely recent, it still needs to be peer reviewed. A paper which gives the technical details of these ideas is in preparation. The dimensional theory is also still in relatively early stages; many issues need to be addressed, such as the detailed description of quantum states and their reduction in terms of this new ontology, its generalization to interacting cases, field theory and the standard model, and the impact on the relation between quantum theory and General relativity (however, see [13][5][14]). Understanding these problems may well require mathematics which does not yet actually exist, but we can take heart, for we humans are blessed with imagination.

7 Conclusion

We have now come to the end of our journey. I made an argument that the freedom to choose one’s axioms coupled with the requirement of consistency should naturally lead us to expect mathematics to be unreasonably effective in modeling reality, but that this unreasonable effectiveness only exists, as it were, as an actualizability until human imagination transforms (parts of) mathematics into an actually effective model of reality. I attempted to illustrate this by showing that using some imagination to formally introduce the very distinction between actualizability and actuality by means of a new axiom into the foundations of mathematics may permit us to understand longstanding seemingly unfathomable phenomena associated with quantum mechanics.

If I succeeded, then you will have gained a whole new appreciation for the role of imagination in mathematics, and along the way may have come to look at quantum mechanics with completely different eyes: Rather than this mysterious if not mystical framework which, as Feynman declared so famously, “nobody understands”, you might now entertain the possibility that it can make perfect sense after all. If so, you will surely agree that if there is anything unreasonable about mathematics, it is this: in mathematics, a little imagination can go a very long way.
where the square bracket includes the terms still subject to taking the limit. The last term is the standard classical free particle action.

Substituting \( \langle x_n | x_{n-1} \rangle = \langle x_n | \hat{U}(\Delta t) | x_{n-1} \rangle \), where \( \hat{U}(\Delta t) = e^{-i\hat{H}_0\Delta t} \) is the time evolution operator, and then inserting a complete set of \( N \) momentum eigenstates \( \sum_{n=1}^{\infty} dp_n \langle p_n | p_n \rangle = 1 \) gives

\[
\lim_{N \to \infty, \Delta t \to 0} \int_{-\infty}^{\infty} dp_N \langle x_N | p_N \rangle \langle p_N | e^{-i\frac{\hat{H}_0}{\hbar}\Delta t} | x_{N-1} \rangle \prod_{n=1}^{N-1} \int_{-\infty}^{\infty} dx_n \int_{-\infty}^{\infty} dp_n \langle x_n | p_n \rangle \langle p_n | e^{-i\frac{\hat{H}_0}{\hbar}\Delta t} | x_{n-1} \rangle
\]

(10)

substitute the free particle Hamiltonian \( \hat{H}_0 = \frac{p^2}{2m} \), expand \( \hat{U}(\Delta t) \) to first order, operate to the left, rewrite the first order eigenvalue expansion as an exponential, use \( \langle x_n | p_n \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{\hat{p}n}{\hbar}} \), and collect the coefficients in the front

\[
\lim_{N \to \infty, \Delta t \to 0} \left( \frac{1}{2\pi\hbar} \right)^N \prod_{n=1}^{N-1} \int_{-\infty}^{\infty} dx_n \int_{-\infty}^{\infty} dp_n \exp \left\{ \sum_{n=1}^{N} \frac{-i\Delta t}{2\hbar} \left( p_n - \frac{m}{\Delta t} (x_n - x_{n-1}) \right) \right\}
\]

(11)

use \( e^{ia_n} \prod_{n=1}^{N-1} e^{ia_n} = e^{\sum_{n=1}^{N} a_n} \) to collect complex exponentials, then complete the square in the exponents

\[
\lim_{N \to \infty, \Delta t \to 0} \left( \frac{1}{2\pi\hbar} \right)^N \prod_{n=1}^{N-1} \int_{-\infty}^{\infty} dx_n \int_{-\infty}^{\infty} dp_n \exp \left\{ \sum_{n=1}^{N} \frac{-i\Delta t}{2\hbar} \left( \frac{m}{\Delta t} \right) (x_n - x_{n-1})^2 \right\}
\]

(12)

where the square brackets are meant to indicate that the product goes to \( N \) for the momentum integrals but only to \( N - 1 \) for the position integrals. Using the Gaussian Formula \( \int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}, a \in \mathbb{C} \), the momentum integrals can be evaluated one by one, and the general formula for \( N \) integrals can be found by induction, giving

\[
\lim_{N \to \infty, \Delta t \to 0} \left( \frac{1}{2\pi\hbar} \right)^N \sqrt{\frac{2m\hbar}{i\Delta t}} \prod_{n=1}^{N-1} \int_{-\infty}^{\infty} dx_n \exp \left\{ \sum_{n=1}^{N} \frac{-i\Delta t}{2\hbar} \left( \frac{m}{\Delta t} \right) (x_n - x_{n-1})^2 \right\}
\]

(13)

which can be rewritten as

\[
\lim_{N \to \infty, \Delta t \to 0} \sqrt{\frac{m}{i2\pi\hbar\Delta t}} \prod_{n=1}^{N-1} \int_{-\infty}^{\infty} dx_n \exp \left\{ \frac{i}{\hbar} \sum_{n=1}^{N} \frac{m}{2} \left( \frac{x_n - x_{n-1}}{\Delta t} \right)^2 \Delta t \right\}
\]

(14)

In the limit \( N \to \infty, \Delta t \to 0 \), the sum in the exponent becomes an integral and its argument a derivative, yielding

\[
\lim_{N \to \infty, \Delta t \to 0} \sqrt{\frac{m}{i2\pi\hbar\Delta t}} \prod_{n=1}^{N-1} \int_{-\infty}^{\infty} dx_n \exp \left\{ \frac{i}{\hbar} \sum_{n=1}^{N} \frac{m}{2} \left( \frac{x_n - x_{n-1}}{\Delta t} \right)^2 \Delta t \right\} = \int_{T}^{T} D[x(t)] e^{i\frac{\pi}{4} \frac{m}{\hbar} \frac{1}{2} \Delta t}
\]

(15)

where the square bracket includes the terms still subject to taking the limit. The last term is the standard symbolic expression for the Feynman path integral in \( 1 + 1 \) dimensions and \( S[x(t)] = \frac{1}{2} \Delta t m \dot{x}(t)^2 / 2 \) is the classical free particle action.
References


[3] Lambert, K. Free Logic and the Concept of Existence Notre Dame J. Form. Logic, 8 133-144


