No-Go Theorems in Physics and Maths: from Bell to Gödel with Hidden Propagator Dynamics

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It is clear physics and mathematics have a special relationship not enjoyed by the rest of science. Although Wigner’s classic title “The Unreasonable Effectiveness of Mathematics in the Natural Sciences” [1] speaks of all science, his claim of unreasonableness is specific to physics. For the rest of science, his unreasonableness claim lies with the structure of maths, such as \( i \), \( e \) and \( \pi \) turning up over the place. But what is so unreasonable about maths’ relationship with physics?

It all started off so well. It can be argued that the identification of a relationship between physics and maths by Galileo defined the birth of physics [2]:

> “Philosophy is written in this grand book – I mean the universe – which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language ... in which it is written. It is written in the language of mathematics ... without which it is humanly impossible to understand a single word of it ...”

This quote contains the seed of the conflict that quantum theory lays bare: is the universe written in maths, or do we just describe the universe using maths? This can be characterised using the territory and map language of Korzybski [3] as:

- Physics realism: physics is the territory and maths is the map
- Maths realism: maths provides the territory and physics is a map

In the first view, maths is a universal language being used to describe reality. So not only can maths be used to describe our reality, it can be used to describe any reality you care to imagine. If you accept paraconsistent logic as maths, then it could even be used to describe magical lands like Terry Prackett’s Discworld [4]. In the universal language view of maths describing reality, the effectiveness of maths can never be unreasonable, no matter how outrageous reality may be.

The maths realism view is of the Platonic realm – where mathematical objects exist – being the true reality which physics glimpses through experiment. To physics realists this smacks of maths mysticism, where the physical world of experiment is some kind of illusion of an underlying mathematical reality. However, every student of physics must find their way to make peace with this, because quantum theory says that in some mysterious way, experimental-reality is not the same as underlying-reality. Is maths “unreasonably effective” or is reality just “unreasonable”?

This clash of views underlies the clash between classical physics – where experimental-reality is reality – and quantum physics. The relationship between maths, physics and reality implied by quantum physics is illustrated in Figure 1. As Smolin points out [5], when maths applies in physics it applies for a reason, which constrains the maths of physics to a maths subset. So if underlying-reality matches a Platonic realm of maths, then it can only be a subspace. But the defining feature in Figure 1 is the distinction between experimental-reality and underlying-reality.

Meta-mathematics provides no-go theorems that define the limits of maths, such as Gödel’s incompleteness theorem [6]. Physics too has its own no-go theorems. Perhaps the most influential has been Bell’s result [7] preventing the physics view of Einstein, Podolsky and Rosen [8] from being realised. But we can also ask whether any maths no-go theorems transfer to physics?
Figure 1: Schematic relationship between mathematics, physics and reality.

The clash between the views of physics realism – represented by EPR – and maths realism – implied by Bell – ignores a third possibility: the maths describing underlying-reality has “issues”. This essay will revisit Bell-EPR [9], but this time with a physical basis for the distinction between experimental-reality and underlying-reality. We still find a no-go theorem, it’s just not Bell’s.

Causality

Any serious definition of physics is based on causality as it provides the basis for the physical laws governing changes in reality. But the way you choose to define causality affects the maths rules of your theory. EPR and Bell critically depend on the terms local and causal, but this can mean different things to different people. For the physics context of Figure 1 such differences can depend on your view of whether experimental-reality is different from underlying-reality.

Here we will define causation propagating through underlying-reality at \( \leq c \) as piecewise causation: a cause on one side of a spatial section of size \( \delta \) will propagate to the other side at \( \leq c \). Continuous causation is then defined by taking the limit \( \delta \rightarrow 0 \). This definition naturally leads to the factorisation condition \( z = xy \) used in Bell-like results.

The significance of using factorisation to express causation is that we get a no-go result for physics theories with the experimental fact of things having a size. For any continuous fields in physics, this constraint requires a norm in the theory. Together with the factorisation condition, this gives the definition of the normed division algebras (NDAs):

\[
\begin{align*}
    z &= x \ast y \\
    \|x\| \|y\| &= \|x \ast y\| \\
    z^2 &= x^2y^2
\end{align*}
\]

This means any theory of physics with continuous causation of continuous fields must be in terms of NDA-valued fields. Unsurprisingly, this is the form of both the Standard Model and General Relativity, but it must also be the form of any theory attempting to unify the two. This can be inverted to a no-go result: if a theory has non-NDA-valued fields, then it’s not physics.

This no-go result says that the search for a theory that unifies the Standard Model and General Relativity is restricting physics to looking for a theory amongst the possible combinations of NDA fields over one another. In this context it is not surprising that this search has led to physicists...
discovering new maths. There are only 4 NDAs – real-numbers, complex-numbers, quaternions and octonions – and they are related to each other: they can be derived in sequence from the previous one by the Cayley-Dickson construction. Physicists have been making discoveries in maths because it they have been looking at the structure of NDA-based maths, and finding this structure limits their possible combinations over one another.

The limits found by this search also suggest the possibility of another no-go type result: physics unification has to involve extra dimensions. But the number of extra dimensions also appears to be restricted. Whether it is string theory or a Kaluza-Klein theory, the number of extra dimensions is found to be restricted in order to yield the Standard Model and General Relativity. The accumulated results of the search for a unified theory are building a picture of NDA fields defining the nature of underlying-reality. If we could imagine physics where underlying-reality could vary dimensions and the dimensionality of fields, then the conditions of continuous causation and norms would pick out the NDAs. The NDAs do seem to define underlying-reality, but how?

Despite the significance of NDA fields in underlying-reality, quantum theory is successful because its predictions are in the discrete terms measured in experiment. The truth is that particles are only ever measured as … well … particles. They are never measured as probability waves. Although the wave property correctly gives particle distributions in experiments, particles are always measured as discrete particles. This discreteness of particle measurements in experimental-reality is a key condition used in Bell’s analysis of whether a classical physics theory can reproduce quantum theory results. Quantum theory conceptually converts NDA fields into natural-number predictions for the discrete particle states of experimental-reality, \( NDA \rightarrow N \). In quantum physics, a discrete initial state is causally changed into a discrete final state via NDA fields, \( N \rightarrow NDA \rightarrow N \). Any theory of this causality of discrete measured events in experimental-reality requires a discrete state transition formulation of causation, distinct from the continuous causation of NDA fields.

We will follow the spirit of EPR and insist a theory should only have terms that directly correspond with “elements of reality”, which for theories over experimental-reality means discrete terms corresponding to the measured discrete particle states. This physically-real notation\(^1\) defines logic values in a theory by whether or not the corresponding discrete state exists in reality. A state transition version of causation will define a causal logic, where changes in the existence of states in reality will directly correspond to logical changes in the discrete maths of the theory. A critical point about this approach to formulating a physics theory is that if it is strictly adhered to, the theory will be mathematically consistent if reality is physically consistent.

**EPR with hidden propagator dynamics**

We now have the concepts to adopt a different approach to the correlated particle spins version of EPR. The underlying issue of quantum theory apparently saying that experimental-reality isn’t the same as underlying-reality will be tackled directly. We will assume that the physical reason for this distinction is hidden propagator dynamics (HPD):

The piecewise propagation of causation is hidden by the timescale of the particle propagation dynamics being a lot less than the timescale of any and all interactions that can be used for the purpose of experimental measurement.

\(^1\) This corresponds to EPR label of “completeness” which isn’t used here to avoid confusion later.
Evidence for HPD is given by the pattern of the running couplings in particle physics. Between the electroweak energy scale and any unification energy scale is a vast “interaction desert”. As energy scale is inversely related to the timescale of the interaction, the crossing of the running couplings says that there exist particle interactions on a timescale far less than the timescale of any interaction that could be used for experimental measurement.

The vast interaction desert allows physics dynamics for the causal propagation of particles to be hidden from experiment. This induces a distinction between underlying-reality – where the particle propagation dynamics occurs – and experimental-reality – where the initial and final states of the dynamics are measured. In simple terms, the distinction is because experiments are just too slow to measure what is happening.

To illustrate the effect of this, imagine a spinning coin where the rotation dynamics is causal and deterministic, but occurs on a timescale much less than any possible physical means to stop the spinning. This difference between the dynamics timescale $t_d$ and the measurement timescale $t_m$ means the coin will continue to rotate during the measurement process. So the coin will be 50/50 heads or tails over the measurement timescale $t_m \gg t_d$ and there is no experimental means to determine the stopping point. Now we can imagine a thought experiment where the coin is stopped in a causally deterministic way, but this will be on the timescale $t_d$ that no experiment can physically realise. So the thought experiment is irrelevant to the results of any physical experiment. HPD induces a distinction between underlying-reality that is not now deterministic. The best an HPD theory over experimental-reality can do is predict the probability of the measured result, because any measurement process realises an average over the hidden dynamics.

In this way HPD can reproduce the same probability over a single measurement as quantum theory – as opposed to a probability over an ensemble – and transfers an aspect of quantum theory that seemed in maths over to physics. However, by itself this is insufficient to change the conclusion of any Bell-type analysis. This is because particle spin is a discrete eigenstate of the full $SU(2)$ rotation group, whereas free objects have rotation group $SO(3)$ and a continuous range of angular momentums. The rotation group issue can be addressed in classical physics by $SU(2)$ being the rotation group for objects that are physically linked, such as in the Dirac belt trick. But what is the physical linkage between particle spins? The experimental fact of discrete spin eigenvalues is even more problematic, but central to Bell-like results.

We could use the “hidden” in HPD to assert the existence of discrete $SU(2)$ spin eigenstates of physically-linked objects in classical physics. It is consistent with HPD and experiment to assert a physical linkage that is hidden from experiment, but leaves the rotation group $SU(2)$ apparent in experimental-reality. The question of a discrete spin eigenstate could be answered with a shrug:
“don’t know, it’s hidden”. This answer is of the same standard as in Bell’s hidden variable theory over experimental-reality, but won’t prove to be as acceptable here.

Figure 2 Spin correlation arrangement of EPR-Bell

Consider the EPR-Bell scenario of a spin singlet state of 2 spin $\frac{1}{2}$ particles. In quantum theory the expectation value for the correlation between the measured spin components of the separated particles along directions $A$ and $B$ is given by:

$$\langle AB \rangle = \langle \psi | \sigma_1 \cdot a \otimes \sigma_2 \cdot b | \psi \rangle = \int \psi(x, t)(\sigma_1 \cdot a \otimes \sigma_2 \cdot b)\psi(x, t)dx = -a \cdot b$$

for the spin singlet state of particles 1 and 2:

$$\psi(x, t) = \varphi(x, t)\chi, \quad \chi = \frac{1}{\sqrt{2}}(\uparrow_1 \downarrow_2 - \downarrow_1 \uparrow_2), \quad \sigma_z \uparrow = \uparrow, \quad \sigma_z \downarrow = -\downarrow$$

This standard form of the calculation gives the misleading impression that spin states are static, and correspond to mathematical objects in a Platonic realm of Hilbert space giving underlying-reality. But this is not what the physics says. Spin is a dynamic state of relativistic rotation in $3+1$ dimensions. The spin up state is a dynamic eigenstate in which the components of spin in the orthogonal directions are not zero, they are just not eigenstates. The superposition of spins is also dynamic, and not static. The expectation value for the spin component of particle 1 in any direction $A$ is zero because in the singlet state the spin is dynamically exploring every possible orientation:

$$\langle A \rangle = \langle \psi | \sigma_1 \cdot a | \psi \rangle = \int \psi(x, t)(\sigma_1 \cdot a)\psi(x, t)dx = 0$$

The spatial domain of both integrals is effectively the surface $S^2$ that the tip of each particle spinor dynamically explores in the spin singlet state. The critical feature that gives the quantum result is the group manifold of $SU(2)$ being $S^3$. The 3-sphere is the closed space in the non-commutative quaternions – a NDA – and a Hopf fibre-bundle where the $S^1$ fibre has a non-trivial twist over the base-space $S^2$. In the spin singlet state the group manifold $S^3$ is being mapped to the spatial sphere $S^2$ that the spin singlet dynamics are being integrated over. It is the topological relationship between $S^3$ and the surface $S^2$ of possible spin orientations that is responsible for the quantum result [10]. The stronger than linear correlation of spins in quantum theory comes from the dynamic integral over $S^2$ of orientations of spin $\frac{1}{2}$ eigenstates of $SU(2)$.
Once this physics behind the quantum result is understood, we can appreciate that any hidden dynamics with all the same features will necessarily reproduce the same integral result for the correlation of spins between A and B. This is what the 3 assumptions outlined above achieve:

1. Hidden propagator dynamics is integrated over the $S^2$ orientations in underlying-reality
2. Physically-linked objects with rotation group $SU(2)$, but where the linkage is “hidden”
3. Discrete $SU(2)$ spin eigenstates are asserted to exist but the physics is “hidden”

So in the EPR-Bell scenario of 2-particle spin correlations these HPD assumptions reproduce the quantum result as they give the same integral result with the same meaning in terms of probability. Furthermore, these HPD assumptions will reproduce every other spin correlation result as quantum theory in every case.

Any suspicion that this might contradict Bell on the grounds that HPD is just a version of Bell’s hidden variable theory is not correct. HPD differs from Bell in rotation group, dynamic integral, and the conditions of causality and determinism. Bell proceeds from experimental results to show that no continuously causal and deterministic theory over experimental-reality can reproduce the same results as quantum theory. HPD can succeed because it doesn’t meet this condition: it is piecewise causal and deterministic over underlying-reality, but an explicit consequence of HPD is that it is not deterministic over experimental-reality. But this doesn’t mean HPD avoids all unpleasantness. It reproduces quantum theory results, and so has exactly the same non-locality issues as quantum theory. These are of the same peculiar variety where the non-locality is not deterministic over experimental-reality, and so cannot be used for super-luminal communication.

Since HPD can reproduce quantum theory, the answer of “it’s hidden” to the spin eigenstate question seems inadequate. So is there a scenario that could give spin eigenstates? Not in 3+1 dimensions. But the attempt to unify the Standard Model and General Relativity is appearing to say that there has to be extra dimensions, and the most plausible explanation for why they aren’t seen in experimental-reality is that they are compactified to the Planck scale. This gives the scale of underlying-reality being far smaller than the scale of experiment – consistent with HPD. In a Kaluza-Klein theory with extra dimensions it is possible for there to exist topological defects in the fabric of the hyperspace-time where space is twisted in a knot around the compactified dimensions [11, 12]. A mapping of the Poincare group to the sphere $S^2$ would give a mapping of the spin group $SU(2)$ to $S^3$, and result in topological spin eigenstates where angular momentum is discrete as it is fixed by the compactification scale. This gives a possible scenario where the physical linkage of classical physics objects with rotation group $SU(2)$ is given by the fabric of hyperspace-time with extra dimensions.

No-go proof for hidden propagator dynamics

However, spin correlations are not the only experimental results that can be used to constrain an HPD theory. There are particle interactions appearing in cloud and bubble chambers showing particle/antiparticle pair creation and destruction. For the possible physics just given, such events would be in terms of the creation and destruction of topological defect/anti-defect pairs. These types of event are seen in solid state physics, where pair creation is driven by the strain energy in the lattice structure of a solid. For the topological defects given above, which must be physically linked to the fabric of the hyperspace-time in order to have rotation group $SU(2)$, the corresponding driver for defect/anti-defect pair creation would be “strain energy” in the fabric of space. So are there experimental results saying the energy of empty space varies with the separation of matter?
The answer is the Casimir effect [13], where energy is released from empty space as 2 pieces of matter are brought closer together [14].

This scenario of topological defects is merely for justification. The proof itself proceeds in the same spirit as Bell’s treatment of hidden variable theories. Since a particle is always measured as a particle, the EPR condition of a theory being in physically-real terms constrains any HPD theory of underlying-reality to be over the natural-numbers of different particle types. The experimental and theoretical modelling conditions applying to any HPD theory over underlying-reality are:

- Piecewise causal and deterministic
- In physically-real terms that are natural-number valued
- Models particle creation and destruction reactions using discrete state transitions
- Conservation rules for discrete particle properties constrain possible state transitions
- Conservation rules for energy and momentum etc. constrain possible state transitions
- Casimir effect and particle energies provide sources of energy for pair creation
- Predicts changes in particle types between initial states and final states

The critical experimental condition applying to this HPD theory is the Casimir effect. As 2 particles get closer, the Casimir effect says there will be more energy available for pair creation. This will give an increasing network of particle creation and destruction patterns for smaller physical scales, in the same way as the networks in Feynman diagrams increase for higher energies in quantum field theory. In fact, the particle reaction networks will be the same in an HPD theory as in quantum field theory because the conservation rules for discrete particle properties constrain them to be the same [11]. The HPD condition says this particle dynamics of pair creation and destruction enabled by the Casimir effect occurs on a timescale less than that of any measurement process. So for an HPD theory over underlying-reality to make predictions for experimental-reality, this hidden dynamics by which an initial particle state propagates to a final particle state must be integrated over. The result of integrating over changes in particle numbers in an HPD theory in discrete natural-number terms, is real-number valued probability predictions over experimental-reality. So as for the EPR-scenario, we find that following Bell’s approach to constrain HPD theories has the surprising result of constraining them to make non-ensemble probability predictions like quantum theory. This analysis of HPD theories agrees with Bell’s analysis of hidden variable theories on the crucial point: no causal and deterministic theory exists over experimental-reality. Einstein’s desire for the probabilities of quantum theory to be banished from physics cannot be realised.

Looking in mathematical detail at any HPD theory over underlying-reality that satisfies these constraints, we find it is axiomatic over particle numbers [11]. Although the theory has continuous values for particle positions and energies, the numbers of particles only changes in discrete events. An effect of modelling this particle physics in strictly physically-real terms is that logic values in the theory are given by whether a particle exists or not, and logical changes in the theory will directly model discrete transitions in particle types. Including the Casimir effect in an HPD theory results in all the operations of arithmetic over the natural-numbers occurring within the terms of the theory itself. This is not the trivial case of the theory being maths and so supports arithmetic. This is an example of something far more interesting, and something that can occur elsewhere in science [11]. Every single arithmetic operation is in terms of a physical process that is being modelled within the theory. This meta-position has the consequence that the theory meets the conditions of Gödel’s incompleteness theorem. So any HPD theory that meets the experimental conditions above is proven to be mathematically incomplete if underlying-reality is consistent [11].
The truth of a necessary trick

In a sense Einstein was right – quantum theory is a trick – but in experimental terms he was wrong – quantum theory is a necessary trick. There seems to be a real distinction between experimental-reality and underlying-reality. Whereas quantum theory implies this distinction mathematically, HPD gives a physical basis for it. Bell-like analysis of HPD theories unexpectedly finds a yes-go result: any HPD theory in physically-real terms is constrained to reproduce quantum theory results in experimental-reality. A plausibility argument was given for a theory that could meet the HPD conditions, but a consequence of meeting them is that this in itself prevents the theory from being detected by experiment, because it will reproduce the same results as quantum theory. In effect an HPD theory is experimentally hidden by quantum theory.

The proof that an HPD theory over underlying-reality can reproduce quantum theory results over experimental-reality implies quantum theory is fundamentally a theory over experimental-reality, and not underlying-reality as has been assumed [15]. This would explain the inability to unify the Standard Model with General Relativity, as you wouldn’t expect it to be possible to unify theories that are over different realities: the Standard Model is over experimental-reality, whereas General Relativity is over underlying-reality. A motivating factor in the maths realism view is the Hilbert space of quantum states looking like the Platonic realm of an underlying maths reality. But this is quantum physics, without the physics. The HPD proof proceeded on the basis of not being fooled by the superficially static form of standard notation, and seeing the dynamics present. It is the effect in experimental-reality of this dynamics which can be reproduced, as long the dynamics is over underlying-reality and is dynamically hidden from experiment.

Although an HPD theory over underlying-reality bypasses Bell’s no-go result by not being deterministic over experimental-reality, it is instead subject to Gödel’s no-go result crossing over from maths to physics. The possibility of a Kaluza-Klein theory giving an HPD theory was discussed because the HPD result implies that quantum theory is really over experimental-reality, and so a unified theory would presumably lie with extending General Relativity over underlying-reality. But any HPD theory that unifies General Relativity and quantum theory is proven to be incomplete, and the suspicion that Gödel’s theorem prevents a complete unified physics is verified. Here, complete is in the maths sense of being able to derive specific predictions of everything in theory. In a sense underlying-reality is “unreasonable”, but only because maths is “unreasonable”.

So where does this leave the relationship between physics and maths? An HPD theory reproducing quantum results looks like physics playing a trick on mathematicians. But then the incompleteness proof for the very same HPD theory looks like maths playing a trick on physicists. But that might not be the end this second trick. The reason the conditions for Gödel incompleteness theorem were met within the terms of an HPD theory lies with the creation of an unlimited network of discrete arithmetic interactions. The same network conditions can occur elsewhere in science [11]. When discrete maths becomes “unreasonable” it might not just be physics that suffers the consequences.
References

[3] The expression "the map is not the territory" appeared in a paper Alfred Korzybski gave at a meeting of the American Association for the Advancement of Science in New Orleans, Louisiana in 1931