Time and Causality in General Relativity

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A **Lorentzian manifold** is a Hausdorff manifold $M$, of dimension $n \geq 2$, endowed with a Lorentzian metric, that is a section $g$ of $T^* M \otimes T^* M$ with signature $(-, +, \ldots, +)$. 
Lorentzian manifolds and light cones

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Light cone

A tangent vector $v \in TM$ is timelike, lightlike, causal or spacelike if $g(v, v) <, =, \leq, > 0$ respectively.
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Time orientation and spacetime

At every point there are two cones of timelike vectors. The Lorentzian manifold is *time orientable* if a continuous choice of one of the cones, termed *future*, can be made. If such a choice has been made the Lorentzian manifold is *time oriented* and is also called *spacetime*. 
Causal Relations

Two events \( p, q \in (M, g) \) are related

- chronologically, \( p \ll q \), if there is a future directed timelike curve from \( p \) to \( q \),

\( I^+ \) is open but \( J^+ \) and \( E^+ \) are not necessarily closed.
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- **chronologically**, $p \ll q$, if there is a future directed timelike curve from $p$ to $q$,
- **causally**, $p \leq q$, if there is a future directed causal curve from $p$ to $q$ or $p = q$, 

 horismotically, $p \rightarrow q$, if there is a maximizing lightlike geodesic segment connecting $p$ to $q$ or $p = q$. 

They can be regarded as relations on $M$, i.e. as subsets of $M \times M$.
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I^+ = \{(p, q) \in M \times M : p \ll q\}, \quad \text{chronology relation}
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$$
J^+ = \{(p, q) \in M \times M : p \leq q\}, \quad \text{causal relation}
$$
$$
E^+ = \{(p, q) \in M \times M : p \rightarrow q\} = J^+ \setminus I^+, \quad \text{horismos relation}
$$

They are all transitive. $I^+$ is open but $J^+$ and $E^+$ are not necessarily closed.
Partially ordered sets

Let $\Delta = \{(p, p) : p \in M\}$

**Preorder**

$R \subseteq M \times M$ is a (reflexive) *preorder* on $M$ if it is

- *reflexive*: $\Delta \subseteq R$,
- *transitive*: $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$,
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A preorder which is

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Total (linear) order

A partial order which is total.

In other words in a total ordering given two points you can decide which one comes before and which one after.
Abstract framework

General Philosophy

Forget about the metric and the conformal structure and work with relations on $M$. 

Strategies

Work with all $I^+$, $J^+$ and $E^+$ and promote their relationships, such as $p \leq q$ and $q \ll r \Rightarrow p \ll r$ to the status of axioms (Kronheimer and Penrose '67). Try to build everything from one single relation (e.g. Causal Set Theory '87), maybe $J^+$. Other candidates? But none of $I^+$, $J^+$ or $E^+$ are both closed and transitive while Seifert's is! 

$J^+ S = \bigcap g' > g J^+ g$. Here $g' > g$ if the light cones of $g$ are everywhere strictly larger than those of $g$. 


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$$J_S^+ = \bigcap_{g' > g} J_g^+.$$  

Here $g' > g$ if the light cones of $g$ are everywhere strictly larger than those of $g$. 
Stable Causality

$(M, g)$ is stably causal if there is $g' > g$ with $(M, g')$ causal.
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Time function (representation)

A *continuous* function \(t : M \rightarrow \mathbb{R}\) such that if \(p < q\) then \(t(p) < t(q)\). (Example Minkowski)
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This causal spacetime does not admit a time function (continuity fails) but it is not stably causal.
Good Properties of $J_S^+$

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Good Properties of $J^+_S$

- $J^+_S$ is a partial order iff $(M, g)$ is stably causal,
- $J^+_S$ is a partial order iff $(M, g)$ admits a time function,
- Under stable causality $J^+_S$ is the smallest closed and transitive relation which contains $J^+$ (i.e. it coincides with $K^+$),
- For every time function $t$ defined the total preorder $T^+[t] = \{(p, q) \in M \times M : t(p) \leq t(q)\}$ and denoted with $A$ the set of time functions on spacetime

$$J^+_S = \bigcap_{t \in A} T^+[t]$$

That is from 'time' one recovers $J^+_S$ not $J^+$! The two relations coincide under strong causality assumptions (causal simplicity).
Conclusions

From causality to time

Stable causality ($J^+ S$) implies the existence of time. This is the analog of Szpilrajn order extension principle: every partial order can be extended to a total order. (But here $T + [t]$ is a total preorder and continuity comes into play!)

From time to causality

In a stably causal spacetime the time functions on spacetime allow us to recover $J^+ S$ (whose antisymmetry is equivalent to stable causality). This is the analog of the result which states that: every partial order is the intersection of the total orders which extend it.

Abstract approach

Time considerations suggest to regard $J^+ S$ (or $K^+ S$) as a natural candidate to build up an abstract causality framework.
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