Time and Causality in General Relativity

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Lorentzian manifolds and light cones

Lorentzian manifolds

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Time orientation and spacetime

At every point there are two cones of timelike vectors. The Lorentzian manifold is *time orientable* if a continuous choice of one of the cones, termed *future*, can be made. If such a choice has been made the Lorentzian manifold is *time oriented* and is also called *spacetime*.

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They can be regarded as relations on M i.e. as subsets of $M \times M$

$I^+ = \{(p,q) \in M \times M : p \ll q\}$,	chronology relation
$J^+ = \{(p,q) \in M \times M : p \leq q\}$,	causal relation
$E^+ = \{(p,q) \in M \times M : p \to q\} = J^+ \backslash I^+$,	horismos relation

They are all transitive. I^+ is open but J^+ and E^+ are not necessarily closed.

Partially ordered sets

Let $\Delta = \{(p, p) : p \in M\}$

Preorder

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Total (linear) order

A partial order which is total.

In other words in a total ordering given two points you can decide which one comes before and which one after.

Abstract framework

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- Work with all I^+, J^+ and E^+ and promote their relationships, such as

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  to the status of axioms (Kronheimer and Penrose '67)
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Other candidates?

But none of I^+, J^+ or E^+ are both closed and transitive while Seifert's is!

$$J_S^+ = \bigcap_{g' > g} J_g^+.$$

Here $g' > g$ if the light cones of g are everywhere strictly larger than those of g.

Time and stable causality

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A *continuous* function $t : M \to \mathbb{R}$ such that if $p < q$ then $t(p) < t(q)$. (Example Minkowski)

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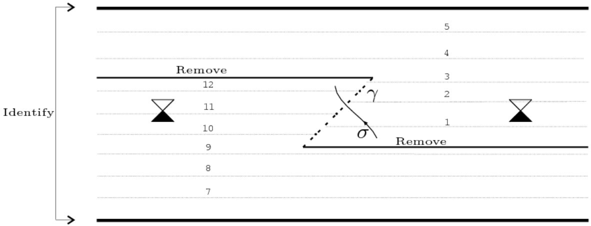
This causal spacetime does not admit a time function (continuity fails) but it is not stably causal.



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- J_S^+ is a partial order iff (M, g) admits a time function,
- Under stable causality J_S^+ is the smallest closed and transitive relation which contains J^+ (i.e. it coincides with K^+),
- For every time function t defined the total preorder $T^+[t] = \{(p, q) \in M \times M : t(p) \leq t(q)\}$ and denoted with A the set of time functions on spacetime

  $$J_S^+ = \bigcap_{t \in A} T^+[t]$$

  That is from 'time' one recovers J_S^+ not J^+! The two relations coincide under strong causality assumptions (causal simplicity).

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Stable causality (antisymmetry of J_S^+) implies the existence of time.

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Abstract approach

Time considerations suggest to regard J_S^+ (or K^+) as a natural candidate to build up an abstract causality framework.