

Notes on unitary/non-unitary structure in gauge theory and gravity

Lawrence B. Crowell

This is an elementary presentation on the relationship between quantum mechanics and gravitation. We start with the examination of the overlap between a state $|\psi(t)\rangle$ and $|\psi(t) + \delta\psi(t)\rangle$. This leads to the expansion

$$\langle\psi|\psi + \delta\psi\rangle = \langle\psi|\psi\rangle + \langle\psi|\frac{\partial\psi}{\partial t}\rangle\delta t + \frac{1}{2}\langle\psi|\frac{\partial^2\psi}{\partial t^2}\rangle\delta t^2$$

With the use of the Schrodinger equation $i|\partial\psi/\partial t\rangle = H|\psi\rangle$ a modulus square of this expansion is then

$$|\langle\psi|\psi + \delta\psi\rangle|^2 = |\langle\psi|\psi\rangle|^2 - (\langle H^2\rangle - \langle H\rangle^2)\delta t^2$$

Physically the term $\sqrt{\langle H\rangle - \langle H\rangle^2} = \Delta E$, which is what defines the Heisenberg uncertainty principle. This also defines a phase

$$\phi = \int dt\sqrt{\langle H\rangle - \langle H\rangle^2} = \int dt\Delta E$$

which is the geometric or Berry phase. For certain systems the above overlap of states can be a measure of the entanglement of states. This is also the Fubini-Study metric for the projective space $CP^n \subset C^{n+1}$. The complex space is projective space C^{n+1} is the $2n + 2$ dimensional state space for a finite dimensional quantum system, where for $n = 0$ this defines the Bloch sphere for a spin system. The complex vector space defines a unitary group $U(n + 1)$, and is an indication of the unitarity of quantum mechanics.

What is of interest is that an elementary example of quantum fields in curved space-time can be defined. Let the energy eigenvalues of the state space be $E_i = \hbar\omega$ which are functions of a one dimensional parameter r , which is a function of time $r = r(t)$. Without worry we will let the spectrum become a continuum, and frequencies a continuous function of this parameter. Let this dependency be a Doppler shift, so the frequency spectrum is $\omega' = (1 - nv/c)\omega$, where n is an index of refraction, $v = dr/dt$ a velocity and c the speed of light. The index of refraction along the one dimensional space is then assumed to vary according to $n = n_0 + \delta n$. The Doppler equation defines a retarded time $1 - nv/c = \omega/\omega' = \nu\tau$ for $\tau = t - r/v$. The effective frequency ν' is then

$$\nu' = \frac{v}{c} \frac{\partial n}{\partial \tau} = \frac{v^2}{c} \frac{\partial \delta n}{\partial \tau}$$

The frequency ν' is then related to a non-Doppler shifted frequency ν by $\nu' = (1 - nv/c)\nu$ for

$$\nu = -\frac{v}{\delta n} \frac{\partial \delta n}{\partial \tau} = -v \frac{\partial \ln(\delta n)}{\partial r}$$

We may then write the Berry phase above from the Fubini-Study metric according to a $\Delta E = (\omega' d\tau - \omega dt)/\hbar$ which according to the frequencies ν , ν' defines the Berry phase

$$\phi = \int_{r_i}^{r_f} \left(1 + \frac{v}{c}\nu\right) = |r_f - r_i| - \frac{v}{c}\ln(n).$$

with the appropriate identification of $vn/c \rightarrow 2GM/rc^2$ the above result reproduces the phase term for a black hole. Further, for the phase to become imaginary the condition is when

$$1 = \frac{\partial \ln(n)}{\partial r}$$

and for the imaginary time identification $t = \hbar/kT$ we find the condition for the onset of an imaginary phase angle as

$$kT = \frac{\hbar\nu}{2\pi}$$

which with the identification between the velocity in a medium with index of refraction recovers the temperature for Hawking radiation

$$T = \frac{\hbar c^3}{8\pi GMk}$$

This simple approach to a derivation of Hawking radiation is based on the notion of a simple bundle fibration over the complex space of quantum states. Here the bundle is just the real line R , which parameterizes the index of refraction along the real line. It is then clear that this program can be extended to more realistic groups which include gravitation.