**FQXi essay contest, 2012 – Questioning the Foundations**
*Which of Our Basic Physical Assumptions Are Wrong?*

# Did God Divide by Zero?

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**Abstract.** It is said that General Relativity fails, because of the occurrence of singularities, and of the non-renormalizability of Quantum Gravity.

Is this failure due to General Relativity, or to our limited understanding? If singularities exist, then did God divide by zero, or we did [1]?

Several fundamental assumptions limited our understanding. Once we get rid of them, we can understand singularities, and see that actually they don't destroy spacetime and information. I show explicitly that the Big-Bang singularity of the FLRW model, and the black hole singularities, can be understood without modifying General Relativity or adding unphysical fields.

Singularities turn out to be our friends. They smoothen and homogenize the Big-Bang. They remove the infinities of the electromagnetic field, and provide a regularization of the quantum fields. They open a door toward a Quantum Gravity, by dimensional regularization.

## 1 Assumptions – wings and prison

If we wait to be 100% certain of our next step, we never move. To do the tiniest step, we have to make some bets – to make assumptions. In science, they are called *hypotheses*.

When we can't solve a problem, we can try to solve a more particular case, under some limiting assumptions. Then we use the experience we gained and move to more general cases. Not despite limitations, but because of them, assumptions help us solve problems.

Assumptions free us from the prison of uncertainty and of a too wide generality. But they become a prison too, when we take them for granted and forget what they really are.

The spacetime in General Relativity apparently breaks down at the singularities. In addition, gravity is non-renormalizable by usual means. I argue that these problems are in fact due to assumptions which are imposed to General Relativity, in particular about the concept of distance. Removing these assumptions removes many of the problems.

Although this is a logical continuation of my essay from the previous FQXi contest [14], it does not depend on it. Since then, the mathematical tools I introduced were successful in showing that:

(1) The FLRW solution extends smoothly beyond the Big-Bang singularity [9].
(2) Using appropriate coordinates, the Schwarzschild [1], and the Reissner-Nordström [2] solutions extend analytically through the singularity.
(3) A consequence of this is that it is possible to have globally hyperbolic spacetimes with black holes whose singularities don't destroy the information [3, 8].
(4) A large class of solutions, including solutions representing inhomogeneous and anisotropic expanding universes, satisfy Penrose's *Weyl curvature hypothesis* [12].
(5) The singularities, instead of destroying the structure of spacetime, cure the infinities of the electromagnetic potential and field [2],
(6) and improve the behavior of QFT and Quantum Gravity at small scales [13, 4].

## 2 Misleading assumptions about space and time

Both Special and General theories of Relativity were born by challenging our fundamental assumptions about space and time. Special Relativity questioned the assumptions that velocities compose by addition, that the speed of light depends on the reference frame, that there is no maximal velocity, our concepts of simultaneity, of causality, of absolute distance and absolute time. General Relativity (GR) challenged the assumptions that space and spacetime should be flat, the assumption that space is a fixed background, which is not affected and does not participate to the phenomena, the idea that you can escape from the gravitational field of any body, at least in principle.

General Relativity continues to question our basic assumptions about space and time. Here are some fundamental assumptions which prevent us from understanding singularities and quantizing gravity.

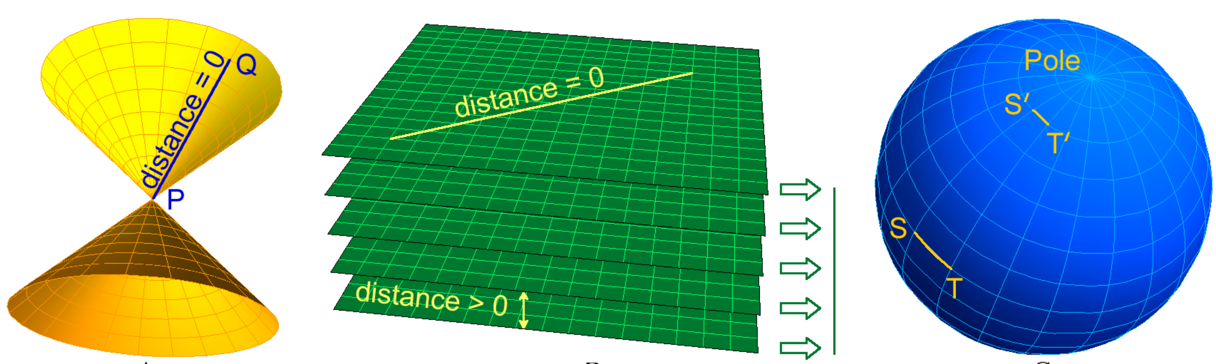
**Assumption 1.** *Singularity theorems predict the breakdown of General Relativity.*

Singularity theorems of Penrose and Hawking predict indeed the occurrence of singularities [15, 16]. But they don't say much about how bad the singularities are. There's no proof that there is no mathematical and physical way to deal with singularities. We will see that there is a way to deal with them, which makes them not only harmless, but even desirable.

**Assumption 2.** *Distance between distinct points can't be zero.*

In fact, even in the relativistic spacetime distinct points may be separated by zero distance (fig. 1 A). But I am talking about a more radical relaxation of the notion of distance: to allow, when necessary, even the distance between distinct points from the same space slice to vanish. This is not a novelty: in Topology one studies spaces which don't carry a notion of distance. Spacetime is firstly a *topological manifold*. The geometric structure, including the notion of distance, is a structure added on top of the topological and differential structures.

Our entire experience is fundamentally rooted in the idea that to go from here to there, you have to move a distance. This is true almost everywhere in the universe. But at singularities, it is possible to have distinct points separated by a distance equal to zero. For example, in fig. 1 B, one can define a distance which between distinct points in the same plane is zero, but between the planes is not zero. Assuming that such points should be coincident identifies each of the planes with a point, and the entire space with a line, and leads to confusion. This kind of identification is responsible for the problems associated to singularities.




A. B. C.

FIGURE 1.

The distance is encoded in the *metric tensor* $g$, which is described at a point $p$ in spacetime, and in a particular frame, as a matrix $g_{ab}(p)$. When that matrix is not invertible, the metric tensor is named *degenerate*, and the distance it determines is zero in certain directions of spacetime [2]. This is a type of singularity.

**Assumption 3.** *Distance is more fundamental than matter.*

One of the main lessons of GR is that the geometry of spacetime is not more fundamental, but it depends on the way matter is arranged. There is an intimate dance between matter and spacetime. The metric tensor, which encodes the notion of distance, contributes to the way matter evolves. Conversely, matter determines the metric. But we forget this so easily, and implicitly think at geometry as a fixed background.

We should allow the evolution equations lead us in finding the metric. If they lead to a singularity, that's it, we have to understand the mathematics and physics of the singularity [3].

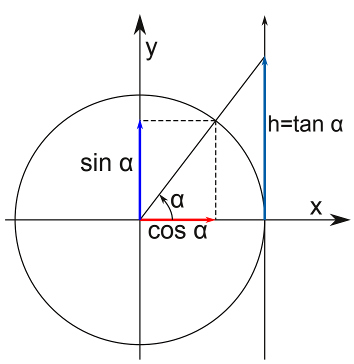
**Assumption 4.** *Trusting the charts.*

Another lesson of GR is that the laws are the same in all coordinate systems (charts). To see this, let's look at the sphere in fig. 1 C. The distance $ST$ is obviously larger than the distance $S'T'$. But if we calculate them in the spherical coordinates, as if the coordinates give the distance, they look of the same length. The laws are indeed independent of the coordinates, with the amendment that when you calculate the distance you use the metric tensor. And when you change the coordinates, the metric tensor changes exactly as needed to give the same distance.

But the spherical coordinates are singular at the poles. When making calculations at the poles, one should use different coordinates, for example defined by another axis of the sphere.

Geometers know how to determine if a coordinate system is singular at a given point, by looking at the metric tensor. This helped for example in showing that the event horizon of a black hole is not a true singularity. But this method fails when the metric tensor is degenerate. We will see that we should not trust the coordinates used so far when dealing with the black hole singularities. They are responsible for making us believe that the flow of matter and information is blocked at singularities, but there are other coordinates which don't do this.

**Assumption 5.** *If the equation fails, it is because the principle it expresses fails.*




A simple example will clarify why this assumption is wrong. Suppose we want the coordinates $x$, $y$, of a point on the unit circle. One can try this in terms of the height $h$, or in terms of the angle $\alpha$. The first approach relies on the formulae $x = \dfrac{1}{1+h^2}$, $y = \dfrac{h}{1+h^2}$, which work for $0 \leq \alpha < \dfrac{\pi}{2}$. But the function tan is singular at $\dfrac{\pi}{2}$. Expressing $x$ and $y$ directly in terms of $\alpha$ avoids this, since $x = \cos\alpha$, $y = \sin\alpha$, which are not singular and extend to all possible values of $\alpha$.

If the equations fail, it doesn't mean that we cannot find other equations to express the same law. Einstein's equations normally fail at singularities. But the same law can be expressed in other equations, equivalent with Einstein's everywhere outside the singularities, but also valid at singularities.

# 3 Through a needle's eye

The Friedmann-Lemaître-Robertson-Walker model describes a homogeneous and isotropic universe. I show that its Big-Bang singularity doesn't break the evolution equations [6, 9].

The FLRW spacetime is obtained by making the size of a symmetric 3D space $\Sigma$ (usually $\Sigma$ is $S^3, \mathbb{R}^3$, or $H^3$) time dependent. This time-dependence is described mathematically by the *warped product* $I \times_a \Sigma$, where the size is given by $a : I \to \mathbb{R}$, $a \geq 0$. The metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\Sigma^2 \tag{1}$$

The energy density $\rho$ and pressure density $p$ are, in terms of the function $a$:

$$\rho = \frac{3}{\kappa}\frac{\dot{a}^2 + k}{a^2}, \quad \rho + 3p = -\frac{6}{\kappa}\frac{\ddot{a}}{a}. \tag{2}$$

They are singular at the Big-Bang, where $a(t) = 0$. But the correct densities are in fact

$$\widetilde{\rho} = \rho\sqrt{-g}, \quad \widetilde{p} = p\sqrt{-g}, \tag{3}$$

and coincide with $p$ and $\rho$ only when expressed in an orthogonal frame, which can't be constructed if the metric is degenerate. Since $\sqrt{-g} = a^3\sqrt{g_\Sigma}$, $\widetilde{\rho}$ and $\widetilde{p}$ are smooth:

$$\widetilde{\rho} = \frac{3}{\kappa}a\left(\dot{a}^2 + k\right)\sqrt{g_\Sigma}, \quad 3\widetilde{p} + \widetilde{\rho} = -\frac{6}{\kappa}a^2\ddot{a}\sqrt{g_\Sigma}. \tag{4}$$

Hence, although $T_{ab}$ is singular, the densitized stress-energy tensor is smooth as well

$$T_{ab}\sqrt{-g} = \left(\widetilde{\rho} + \widetilde{p}\right)u_a u_b + \widetilde{p}g_{ab}. \tag{5}$$

It is natural to use instead of the stress-energy tensor its densitized version, since it is the one obtained from the Lagrangian density [4]. From the Lagrangian formulation one obtaines the Einstein equation by dividing its densitized version by $\sqrt{-g}$, which is not allowed when $\sqrt{-g} = 0$. So we use instead the densitized Einstein's equation [5]

$$G_{ab}\sqrt{-g} = \kappa T_{ab}\sqrt{-g}, \tag{6}$$

which is smooth, and can describe the flow through the Big-Bang singularity.

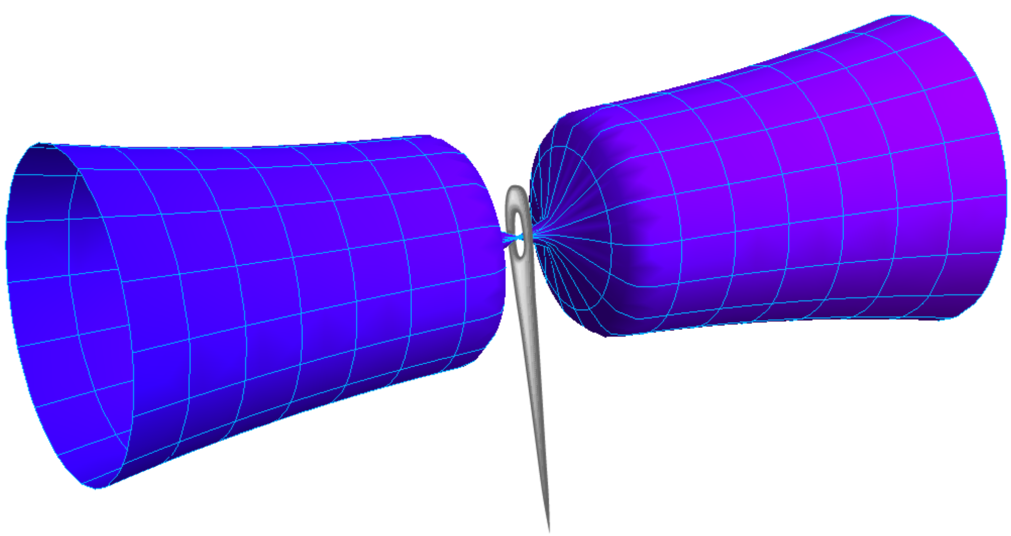


FIGURE 2. The equations extend smoothly through the Big-Bang singularity.

# 4 Taming the singularities

The FLRW Big-Bang singularity is smooth, and it is due to the fact that the metric becomes degenerate – we say that it is *benign*. If the singularity is due to the fact that some of the components of the metric are divergent ($g_{ab} \to \infty$), we call it *malign* (fig. 3).

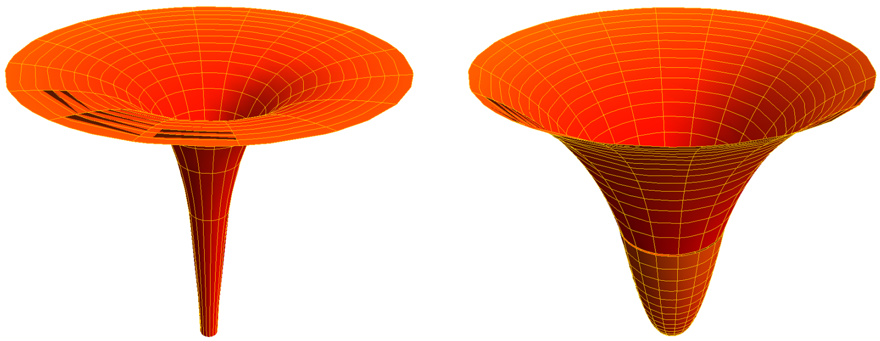



FIGURE 3. A. Malign singularity, $g_{ab} \to \infty$. B. Benign singularity, $\det g \to 0$ smoothly.

Apparently, the stationary black hole singularities are malign. But we can find transformations of coordinates which make them smooth. As a bonus, for charged black holes, also the electromagnetic potential and field are smoothened and bounded.

## 4.1 Schwarzschild black hole

The Schwarzschild metric is, in Schwarzschild coordinates:

$$\mathrm{d}s^2 = -\frac{r-2m}{r}\mathrm{d}t^2 + \frac{r}{r-2m}\mathrm{d}r^2 + r^2\mathrm{d}\sigma^2, \tag{7}$$

where $\mathrm{d}\sigma^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$. The singularity at $r=0$ appears malign, because $\dfrac{r-2m}{r} \to \infty$.

In non-singular coordinates $(r,t) \mapsto (\tau^2, \xi\tau^4)$ I proposed in [1], it becomes smooth at $r=0$:

$$\mathrm{d}s^2 = -\frac{4\tau^4}{2m-\tau^2}\mathrm{d}\tau^2 + (2m-\tau^2)\tau^4\left(4\xi\mathrm{d}\tau + \tau\mathrm{d}\xi\right)^2 + \tau^4\mathrm{d}\sigma^2. \tag{8}$$

## 4.2 Reissner-Nordström black hole

The metric of a charged non-rotating black hole is, in Reissner-Nordström coordinates

$$\mathrm{d}s^2 = -\frac{\Delta}{r^2}\mathrm{d}t^2 + \frac{r^2}{\Delta}\mathrm{d}r^2 + r^2\mathrm{d}\sigma^2 \tag{9}$$

where $\Delta = r^2 - 2mr + q^2$. It is singular at $r=0$, as is the electromagnetic potential:

$$A = -\frac{q}{r}\mathrm{d}t. \tag{10}$$

In non-singular coordinates $(t,r) \mapsto (\tau\rho^T, \rho^S)$ I introduced in [2], the metric is smooth:

$$\mathrm{d}s^2 = -\Delta\rho^{2T-2S-2}\left(\rho\mathrm{d}\tau + T\tau\mathrm{d}\rho\right)^2 + \frac{S^2}{\Delta}\rho^{4S-2}\mathrm{d}\rho^2 + \rho^{2S}\mathrm{d}\sigma^2, \tag{11}$$

where $T > S \geq 1$. The electromagnetic potential is non-singular:

$$A = -q\rho^{T-S-1}\left(\rho\mathrm{d}\tau + T\tau\mathrm{d}\rho\right). \tag{12}$$

# 5 Beyond the end of time

To avoid the problem of singularities, Roger Penrose proposes the *cosmic censorship hypothesis* – stating that singularities should be either in the past at the beginning of time (like the Big-Bang), or in the future at the end of time (like the Big-Crunch), or hidden behind an event horizon from which nothing can escape [17, 18]. There is a lot of work to be done in proving this conjecture, or checking in what cases holds (it would take about 100 years, according to S. Klainerman). But if the black holes evaporate, revealing for an instant a naked singularity (fig. 4 A), can we still hope that the cosmic censorship conjecture is true, or should we revise it?

What happens with the *information*, the data describing the fields at a given time, when matter collapses to form a black hole, which subsequently evaporates? Apparently the *information is lost* in the singularity (fig. 4 A). The problem is that, when taking into consideration Quantum Mechanics, the information loss causes in particular a *violation of unitarity*, if a system which is entangled with another system is lost in the singularity [19, 20].

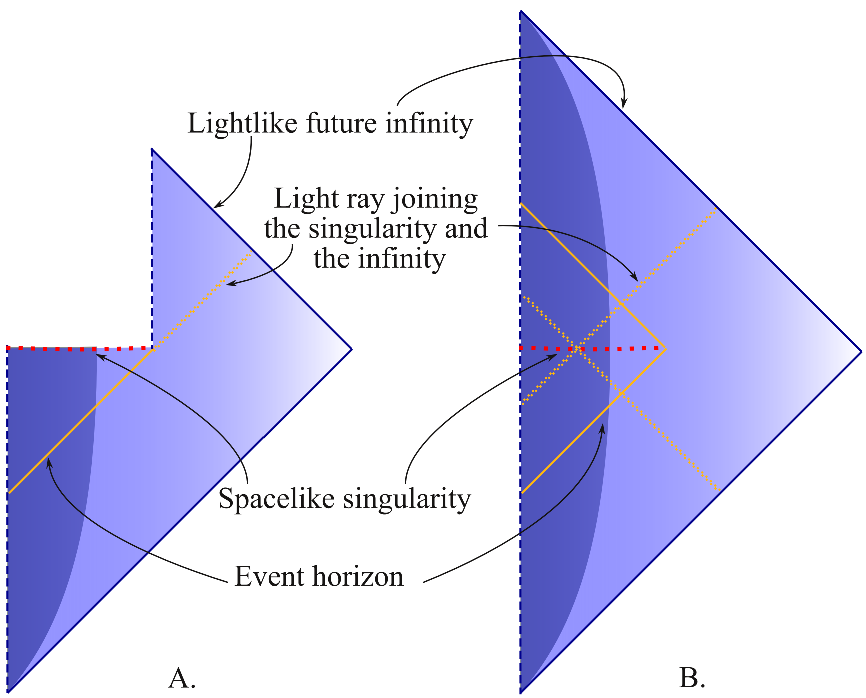



Figure 4. Penrose-Carter diagrams for evaporating black holes: A. when the singularity is malign, B. when the singularity is benign

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But if the singularity is benign, as it is when expressed in non-singular coordinates as in section §4, the field equations can be extended beyond the singularity, and the information is preserved (fig. 4 B) [1, 2, 7, 8, 3].

If the singularities are not as harmful as initially thought, then why censoring them? We can use the fact that the black hole solutions can be extended analytically at the singularities, and construct globally hyperbolic spacetimes with singularities, as I did in [8, 3].

# 6 Weyl curvature hypothesis and the arrow of time

Space seems not to have a preferred direction, but time obviously has, since it distinguishes between past and future. We remember the past, and not the future. One can't unscramble eggs. The Big-Bang is at a particular edge of time, and not at the other. The puzzle of the *arrow of time* is one of the most intriguing ones.

The problem is not that the entropy tends to increase toward a maximum, this was already explained by Boltzmann. The complementary question is what happened at the Big-Bang, that gave birth to the arrow of time?

According to Penrose, the answer is that the Big-Bang was very, very homogeneous and isotropic, with very low entropy. His estimates showed that only a small part of the possible initial conditions could satisfy this condition: one in $10^{10^{123}}$ (for comparison, the number of particles in the visible universe is of just about $10^{90}$). He considers responsible for this homogeneity and isotropy the Weyl curvature tensor, so this has to be very small at the Big-Bang (the *Weyl curvature hypothesis*) [21].

But what is the Weyl curvature? The Riemann curvature tensor encodes all the information about the way spacetime is curved. Part of it is responsible for curving the spacetime in a way which uniformly shrinks the volumes. This involves the Ricci tensor $R_{ab} = R^s{}_{asb}$, and the scalar curvature $R = R_s{}^s$. The remaining part, the *Weyl curvature tensor* $C_{abcd}$, encodes the tidal effects, which preserve the volume. This is the decomposition:

$$R_{abcd} = \underbrace{\frac{1}{12} R (g \circ g)_{abcd} + \frac{1}{2} (S \circ g)_{abcd}}_{\text{Ricci part}} + \underbrace{C_{abcd}}_{\text{Weyl part}} \tag{13}$$

where $S_{ab} := R_{ab} - \frac{1}{4} R g_{ab}$, and the product denoted by $\circ$ is defined by

$$(h \circ k)_{abcd} := h_{ac} k_{bd} - h_{ad} k_{bc} + h_{bd} k_{ac} - h_{bc} k_{ad}. \tag{14}$$

There is a large class of benign singularities, named *semi-regular*. Spacetimes having only semi-regular singularities are also named semi-regular. On a semi-regular spacetime the Riemann curvature $R_{abcd}$ is smooth. This can't ensure the smoothness of the Ricci and scalar curvatures (needed for the Einstein equation). Fortunately we can write a densitized version of Einstein's equation, equivalent with it at non-singular points, but which in addition works at singularities.

An even special class of semi-regular spacetimes, named *quasi-regular*, are those whose Ricci decomposition (13) is smooth [11]. At a quasi-regular Big-Bang singularity, $C_{abcd}$ is smooth. Normally, the Riemann curvature lives in a space constructed from the cotangent space $TM$. But at the semi-regular singularities, it lives in a space constructed from a subspace of the cotangent space $T^{\bullet}M := \flat(TM) < T^{*}M$, where $(\flat(X))(Y) := g(X, Y)$ (see note 2). The Weyl curvature lives in this space too. But the only tensor with the symmetries of the Weyl curvature which lives in dimension lower than 4 is equal to 0. Hence the Weyl curvature vanishes at singularities [12].

Therefore, the quasi-regular singularities automatically satisfy the Weyl curvature hypothesis. There are many examples of such singularities, including the ones obtained by rescaling a regular metric, and the Schwarzschild singularity as expressed in §4.1. Even the FLRW singularity is like this, but this case is trivial, because $C_{abcd} = 0$ everywhere.

In [12] I showed that even more general cosmological models, which are inhomogeneous and anisotropic, satisfy the Weyl curvature hypothesis in a non-trivial way.

# 7 Quantum Gravity – a marriage made in Flatland

Not only General Relativity is confronted with infinities, but also Quantum Field Theory. The calculations in QFT are plagued with infinities, and to obtain finite results, one applies techniques of regularization and renormalization.

These infinities go away automatically in spacetimes of lower dimension. That's why one method of regularization is to work in "almost 4D", *i.e.* in 4D$-\varepsilon$, and take the limit $\varepsilon \to 0$.

Non-renormalizable theories like Quantum Gravity also work fine in lower dimension. Because the Weyl curvature tensor vanishes, in lower dimension there are no local degrees of freedom like gravitons, hence no non-renormalizability problems [22]. Different results suggest that a dimension $< 4$ may act like a dimensional regulator for QFT and for QG (see [23, 13, 4] and references therein).

At benign singularities, spacetime undergoes a dimensional reduction, hence they may be the needed dimensional regularizers for QG. In the following I show some ways in which dimensional reduction manifests at benign singularities, and its effects [13, 4].

• At the points $p$ where the metric becomes degenerate, the rank of the metric is reduced, and a geometric, or *metric reduction* takes place:

$$\operatorname{rank} g_p = \dim T_{p\bullet}M = \dim T_p{}^\bullet M < 4. \tag{15}$$

The spaces $T_{p\bullet}M := T_pM/\ker \flat_p$ and $T_p{}^\bullet M := \operatorname{im} \flat_p$ are associated to the tangent space (see note 2). The spacetime fields live in these lower-dimensional spaces (or tensor products of them).

• If the metric is degenerate on a 4D region, the spacetime in that region is isometric with a lower-dimensional spacetime [6]. This suggests a strong connection with the *topological dimensional reduction* researched by D.V. Shirkov and P. Fiziev, which removes the singularity of the coupling constant at least for the Klein-Gordon field [24, 25, 26, 27].

• To make non-renormalizabile theories like Quantum Gravity renormalizable, G. Calcagni proposes to make the measure $\varrho(x)$ from the Lagrangian scale dependent [28]:

$$S = \int_{\mathcal{M}} \mathrm{d}\varrho(x)\, \mathcal{L} \tag{16}$$

He proposed for instance that the measure is expressed in terms of functions $f_{(a)}(x)$, so that some of them vanish while the scale goes to zero:

$$\mathrm{d}\varrho(x) = \prod_{a=0}^{D-1} f_{(a)}(x)\, \mathrm{d}x^a \tag{17}$$

To acquire this scale dependence of the measure, he developed the *fractal universe* theory.

In the approach presented in this essay, this reduction of the measure happens naturally near the points where the metric is degenerate, since:

$$\mathrm{d}\varrho(x) = \sqrt{-\det g}\mathrm{d}x^D \to 0, \tag{18}$$

and when the metric is diagonal in some coordinates $(x^a)_{a=0}^D$, it even has the form (17), with

$$f_{(a)}(x) = \sqrt{|g_{aa}(x)|}. \tag{19}$$

• In subsection §4.2, equation (11), the Reissner-Nordström metric is made analytic at the singularity [2]. This solution can be used to model classical *charged particles*. The metric reduces its dimension to dim $= 2$. These coordinates depend on two numbers $S$ and $T$, and to admit a space+time foliation, they have to satisfy $T \geq 3S$. Is this anisotropy connected to *Hořava-Lifschitz* gravity [29], which relies on a similar anisotropy between space and time to obtain renormalizability?

The quantities $\det g$ and $C_{abcd}$ vanish as approaching singularities. But do they decrease with the scale, as Quantum Gravity needs? The following argument seems to support this conjecture. As the energy approaches the high energy limit, the number of the particles per volume in the Feynman diagrams increases. If particles are benign singularities, this means that $\det g \to 0$ and $C_{abcd} \to 0$ as the energy goes higher. This may give the needed regularization [13].

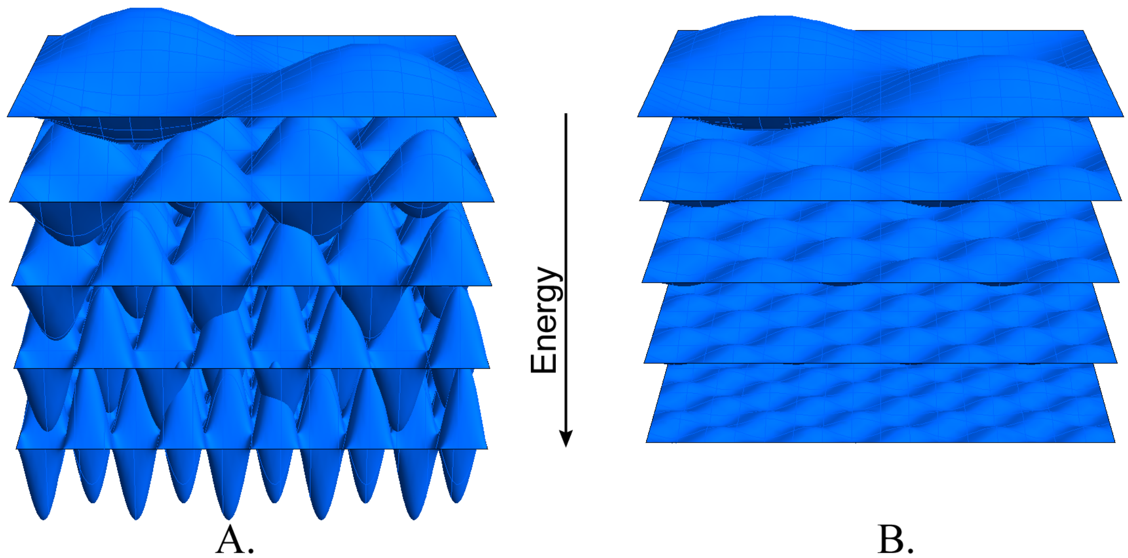


FIGURE 5. Behavior of fields at lower scales. A. If one assumes a non-degenerate metric, quantum fluctuations appear to become infinite as the scale tends to zero. B. *Conjecture:* Singularities act as dimensional regularizers for Quantum Gravity.

# Did God divide by zero?

Science is about making assumptions and continuously challenging them. We learn new paradigms, but periodically we need to be ready to unlearn them too. We have to be aware of the guesses we made on the way, if we want to be able to go back easily and replace them with better hypotheses.

If General Relativity is a faithful description of reality, then apparently "God divided by zero" at singularities. In this essay I showed that in fact we are those who divided by zero. By not dividing by $\sqrt{-g}$ in §3, we could see that the Big-Bang singularities can be freed of infinities. In the Schwarzschild and Reissner-Nordström solutions, the coordinates employed introduce division by $r$ or $r^2$, where $r \to 0$. But the non-singular coordinates I proposed in §4 avoid dividing by 0. This shows that the division by zero was due only to our assumptions, and not to General Relativity or God.

While usually it is claimed that solving quantization will lead to an avoidance of singularities, in this essay I presented the less explored opposite alternative: we can solve the problem of singularities, and this helps us solving the problem of Quantum Gravity. There is still a long way, since the research in this direction is just beginning.

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# Notes

1 "Black holes are where God divided by zero." I think Steven Wright said this.

2 The elementary properties of degenerate metrics are represented in fig. 6.

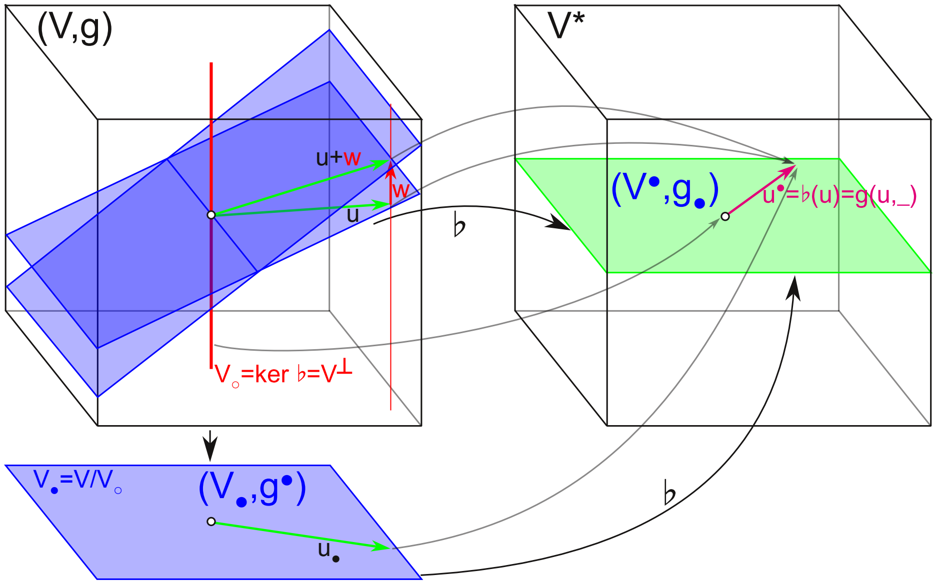


FIGURE 6. The geometry of a degenerate metric.

$(V, g)$ is an inner product vector space (fig. 6). The morphism $\flat : V \to V^*$ is defined by $u \mapsto u^\bullet := \flat(u) = u^\flat = g(u, \_)$. The radical $V_\circ := \ker \flat = V^\perp$ is the set of isotropic vectors in $V$. $V^\bullet := \operatorname{im} \flat \leq V^*$ is the image of $\flat$. The inner product $g$ induces on $V^\bullet$ an inner product defined by $g_\bullet(u_1^\flat, u_1^\flat) := g(u_1, u_2)$, which is the inverse of $g$ iff $\det g \neq 0$. The quotient $V_\bullet := V/V_\circ$ consists in the equivalence classes of the form $u + V_\circ$. On $V_\bullet$, $g$ induces an inner product $g^\bullet(u_1 + V_\circ, u_2 + V_\circ) := g(u_1, u_2)$.

Once we have defined the reciprocal inner product $g_\bullet(\omega, \tau)$, we can define covariant contraction for 1-forms from $V^\bullet$. Then we define it for tensors with two covariant indices living in $V^\bullet$, by $T(\omega_1, \ldots, {}_\bullet, \ldots, {}_\bullet, \ldots, \omega_k)$.

3 *Singular semi-Riemannian geometry* [5, 6] introduces generalizations of semi-Riemannian spacetimes which admit degenerate metric but have smooth Riemann curvature $R_{abcd}$. A *singular semi-Riemannian manifold* $(M, g)$ is a differentiable manifold $M$ with a symmetric bilinear form $g$ (which may become degenerate), named *metric*, on the tangent bundle $TM$. *Singular General Relativity* is the application of the methods of singular semi-Riemannian geometry to General Relativity.

The main problem are the geometric quantities which can't be defined or are singular:

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cs} (\partial_a g_{bs} + \partial_b g_{sa} - \partial_s g_{ab})$$

$$R^d{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^d{}_{bs} \Gamma^s{}_{ac} - \Gamma^d{}_{cs} \Gamma^s{}_{ab}$$

$$R_{ab} = R^s{}_{asb}, \;\; R = g^{pq} R_{pq}$$

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$$

They can't be normally defined even for finite $g_{ab}$ if $\det g \to 0$, since then $g^{ab} \to \infty$. The solution is to use non-singular quantities, equivalent to the singular ones for non-degenerate metric $g$ (table 1). These non-singular quantities are constructed in [5, 6, 11].

| Singular | Non-Singular | When g is... | Non-Singular | When g is... |
|---|---|---|---|---|
| $\Gamma^{c}{}_{ab}$ (2nd) | $\Gamma_{abc}$ (1st) | smooth | | |
| $R^{d}{}_{abc}$ | $R_{abcd}$ | semi-regular | | |
| $R_{ab}$ | $R_{ab}\sqrt{\lvert\det g\rvert}^{W}$, $W \leq 2$ | semi-regular | $\mathrm{Ric} \circ g$ | quasi-regular |
| $R$ | $R\sqrt{\lvert\det g\rvert}^{W}$, $W \leq 2$ | semi-regular | $Rg \circ g$ | quasi-regular |

Table 1.

The *covariant derivative* $\nabla_X Y$ can't be defined. We can use instead the *Koszul form*:

$$\begin{aligned} \mathcal{K}(X,Y,Z) \ := \ & \frac{1}{2}\{X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ & -\langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle\}. \end{aligned} \tag{20}$$

The Christoffel's symbols of the first kind are thus $\Gamma_{abc} = \mathcal{K}(\partial_a, \partial_b, \partial_c)$.
For non-degenerate metric $\nabla_X Y = \mathcal{K}(X,Y,\_)^{\sharp}$. For the degenerate case, we use instead the *lower covariant derivative*, defined as $(\nabla^{\flat}_X Y)(Z) := \mathcal{K}(X,Y,Z)$, $\forall Z \in \mathfrak{X}(M)$.
The contraction between covariat indices, which normally requires the inverse of the metric, can be defined even when the inverse is not defined, as explained in Note 2. We extend this definition to the tensor fields on a singular semi-Riemannian manifold. We can use it to define the *covariant derivative* for differential forms:

$$(\nabla_X \omega)(Y) := X(\omega(Y)) - g_{\bullet}(\nabla^{\flat}_X Y, \omega) \tag{21}$$

A singular semi-Riemannian manifold is named *radical-stationary* if $\mathcal{K}(X,Y,\_) \in \mathcal{A}^{\bullet}(M) := \Gamma(T^{\bullet}M)$. If in addition $\nabla_X \nabla^{\flat}_Y Z$ is smooth, the manifold is called *semi-regular*.
On a radical-stationary manifold one can define the *Riemann curvature tensor*:

$$R(X,Y,Z,T) := (\nabla_X \nabla^{\flat}_Y Z - \nabla_Y \nabla^{\flat}_X Z - \nabla^{\flat}_{[X,Y]} Z)(T) \tag{22}$$

$$R_{abcd} = \partial_a \Gamma_{bcd} - \partial_b \Gamma_{acd} + (\Gamma_{ac\bullet}\Gamma_{bd\bullet} - \Gamma_{bc\bullet}\Gamma_{ad\bullet}) \tag{23}$$

The Ricci tensor is $\mathrm{Ric}(X,Y) := R(X,{}_{\bullet},Y,{}_{\bullet})$. The scalar curvature is $s := \mathrm{Ric}({}_{\bullet},{}_{\bullet})$.
If $(M,g)$ is semi-regular, the Riemann curvature is a smooth tensor field, but the Ricci and scalar curvatures may be singular. But in four dimensions we can write a densitized version of the Einstein equation, which for regular metric it is equivalent to Einstein's equation, but works at singularities too [5]:

$$G_{ab}\sqrt{-\det g}^{W} + \Lambda g_{ab}\sqrt{-\det g}^{W} = \kappa T_{ab}\sqrt{-\det g}^{W}, W \leq 2. \tag{24}$$

4 The densitized stress-energy tensor is obtained from the Lagrangian density by

$$T^{ab}\sqrt{-g} = -2\frac{\delta(\mathcal{L}_M\sqrt{-g})}{\delta g_{ab}}. \tag{25}$$

5 The case containing a positive cosmological constant $\Lambda > 0$ can be reduced to the case $\Lambda = 0$ presented here, by redefining $\rho$ and $p$.

6 This follows from a theorem of Kupeli [30]: for constant signature, the space is locally a *warped product* $M = P \times_0 N$ between lower dimensional manifolds, a manifold $P$ and a manifold $N$ with metric 0.